Introduction to Machine Learning CMU-10701

2. Basic Statistics

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Remember the color coding

Important

Not so important

You can sleep now…

Please ask ***Questions***
and give us ***Feedbacks***!

2. Basic Statistics

Essential tools for data analysis

Outline

Theory:

- Probabilities:
  - Probability measures, events, random variables, conditional probabilities, dependence, expectations, etc
- Bayes rule
- Parameter estimation:
  - Maximum Likelihood Estimation (MLE)
  - Maximum a Posteriori (MAP)

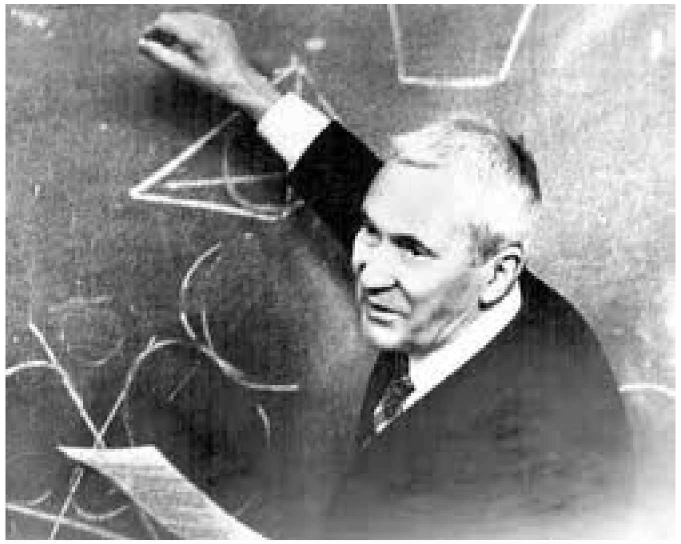
Application:

Naive Bayes Classifier for
- Spam filtering
- “Mind reading” = fMRI data processing

What is the probability?

Probabilities

Bayes



Kolmogorov

Probability

- Sample space, Events, σ-Algebras
- Axioms of probability, probability measures
  - What defines a reasonable theory of uncertainty?
- Random variables:
  - discrete, continuous random variables
- Joint probability distribution
- Conditional probabilities
- Expectations
- Independence, Conditional independence

Sample space

Def: A ***sample space*** Ω is the set of all possible outcomes of a (conceptual or physical) random experiment. (Ω can be finite or infinite.)

Examples*:*

–Ω may be the set of all possible outcomes of a dice roll (1,2,3,4,5,6)

-Pages of a book opened randomly. (1-157)

-Real numbers for temperature, location, time, etc

Events

We will ask the question:

What is the probability of a particular event?

Def: *Event* *A is a* ***subset*** *of the sample space* Ω

Examples:

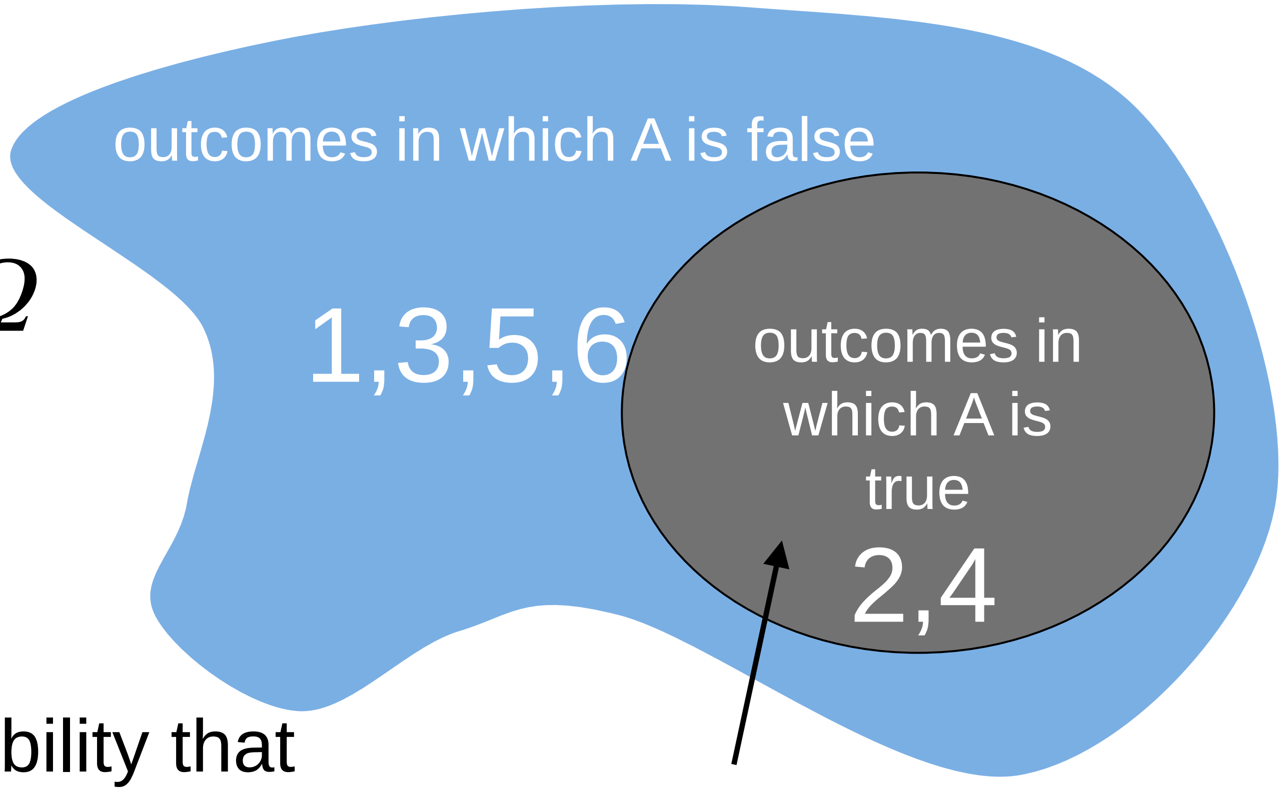
What is the probability of

- *the book is open at an odd number*
- *rolling a dice the number* <4
- *a random person's height X :* $a<X<b$

Probability

Def: *Probability P(A), the probability that event (subset) A happens,* is a function that maps the event *A* onto the interval [0, 1]. *P(A)* is also called the **probability measure** of *A*.

sample space Ω



Example: What is the probability that the number on the dice is 2 or 4?

P(A) is the volume of the area.

What defines a reasonable theory of uncertainty?

Kolmogorov Axioms

(i) Nonnegativity: $P(A) \geq 0$ for each A event.

(ii) $P(\Omega) = 1$.

(iii) σ-additivity: For disjoint sets (events) A_i, we have

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

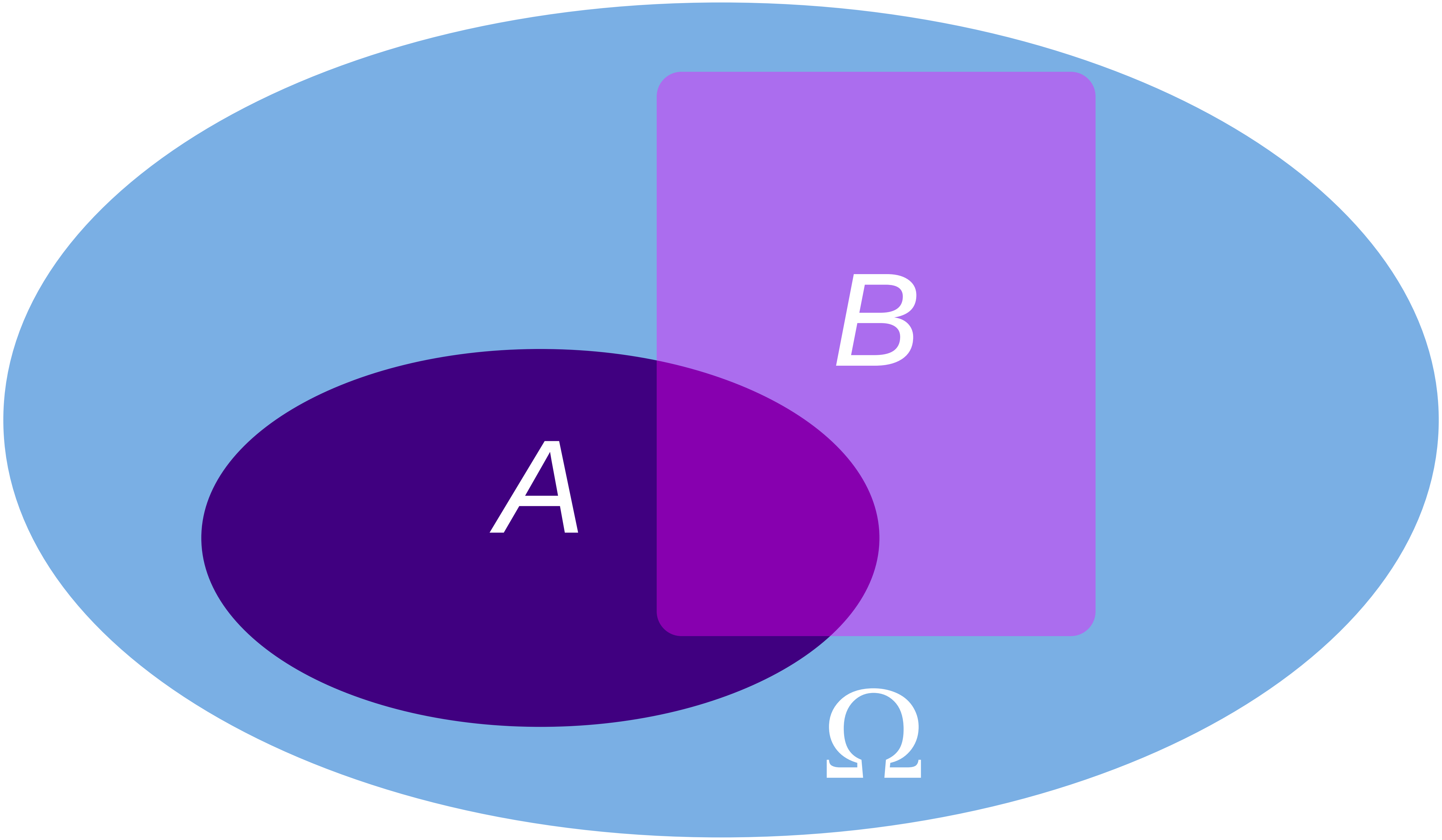
Consequences:

$P(\emptyset) = 0$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$P(A^c) = 1 - P(A)$.

Venn Diagram



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Random Variables

Def: Real valued **random variable** is a function of the outcome of a randomized experiment

$$X : \Omega \to \mathbb{R}$$

$$P(a < X < b) \doteq P(\omega : a < X(\omega) < b)$$
$$P(X = a) \doteq P(\omega : X(\omega) = a)$$

Examples:

- **Discrete random variable examples (Ω is discrete):**
- $X(\omega)$ = True if a randomly drawn person (ω) from our class (Ω) is female
- $X(\omega)$ = The hometown $X(\omega)$ of a randomly drawn person (ω) from our class (Ω)

Random Variables

Sometimes Ω can be quite abstract

$$\Omega = [0, \infty) \times \{1, \ldots, 145\}$$

$$\omega = (\omega_1, \omega_2) \in \Omega$$

Continuous random variable:

Let $X(\omega_1,\omega_2)= \omega_1$ be the heart rate of a randomly drawn person ($\omega=\omega_1,\omega_2$) in our class Ω

$$P(a < X < b) \doteq P((\omega_1, \omega_2) : a < X(\omega_1, \omega_2) < b)$$

What discrete distributions do we know?

Discrete Distributions

- Bernoulli distribution: Ber(p)

$\Omega = \{\text{head, tail}\}\; X(head) = 1,\; X(tail) = 0.$

$$P(X = a) = P(\omega : X(\omega) = a) = \begin{cases} p, & \text{for } a = 1 \\ 1 - p, & \text{for } a = 0 \end{cases}$$

- Binomial distribution: Bin(n,p)

Suppose a coin with head prob. *p* is tossed *n* times. What is the probability of getting *k* heads and *n-k* tails?

$\Omega = \{\text{ possible } n \text{ long head/tail series}\},\ |\Omega| = 2^n$

$K(\omega) = \text{number of heads in } \omega = (\omega_1, \ldots, \omega_n) \in \{\text{head, tail}\}^n = \Omega$

$$P(K = k) = P(\omega : K(\omega) = k) = \sum_{\omega : K(\omega) = k} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

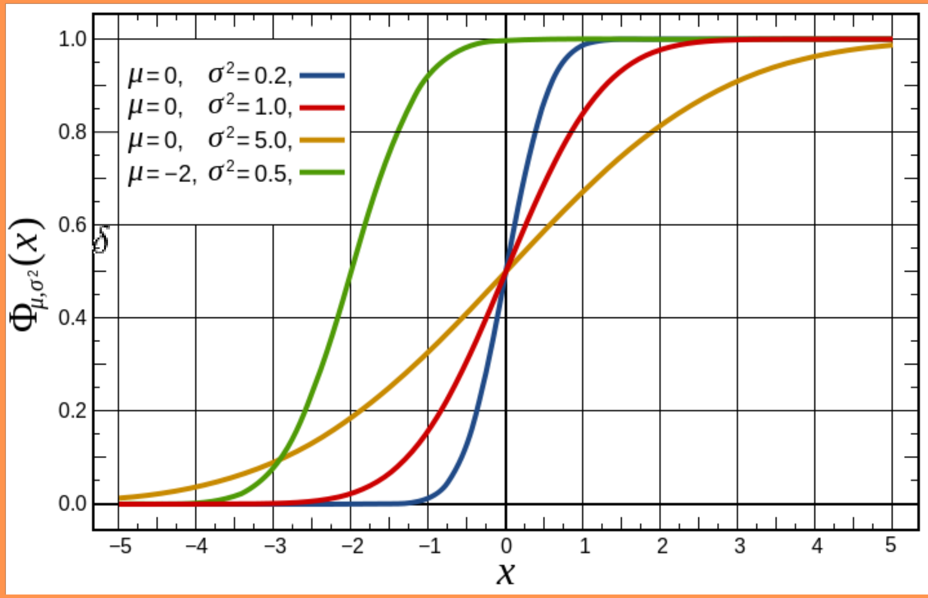
Continuous Distribution

Def: continuous probability distribution: its **cumulative distribution function** is **absolutely continuous**.

Def: cumulative distribution function

USA: $$F_X(z) = P(X \leq z)$$

Hungary: $$F_X(z) = P(X < z)$$



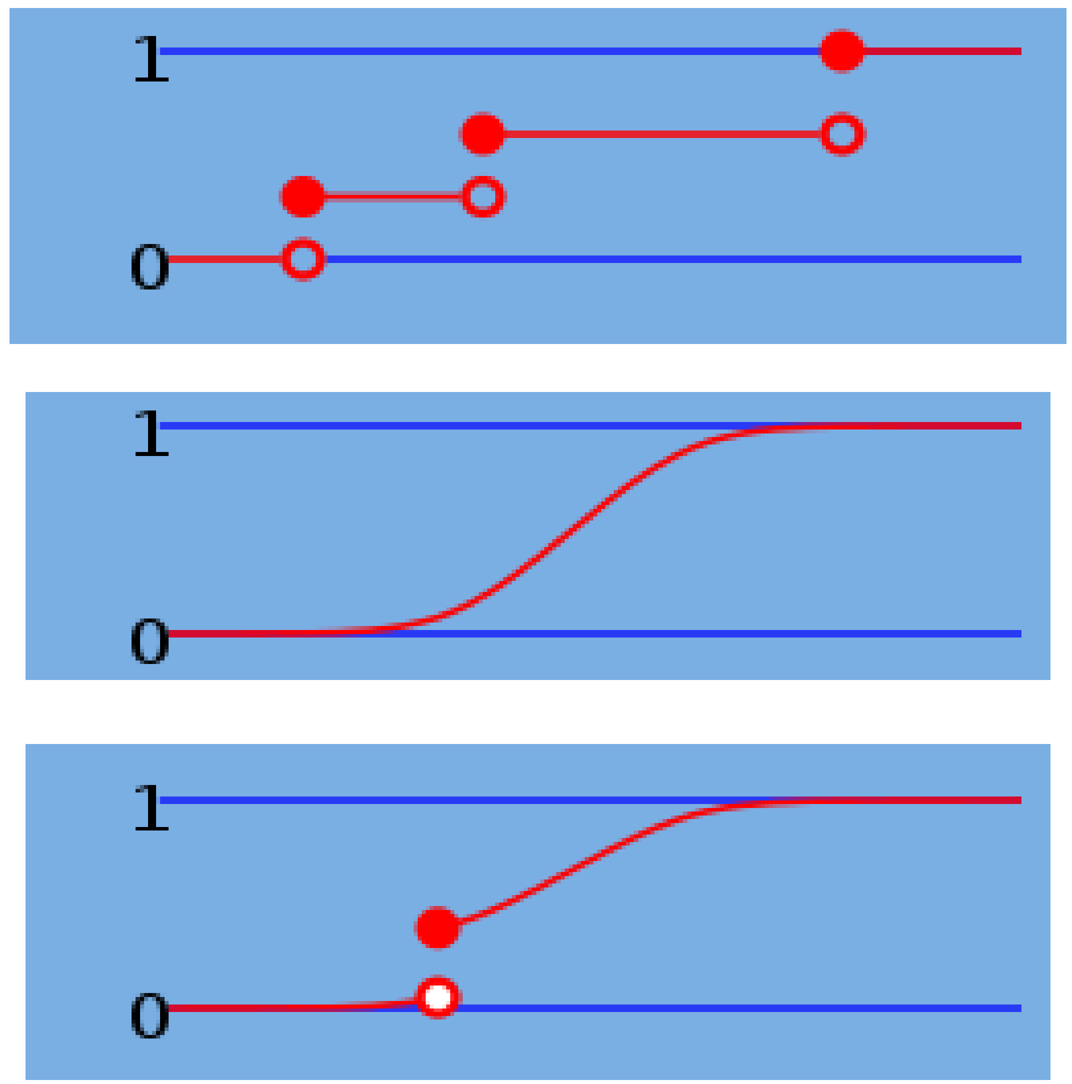
Def : Let $F(-\infty) = 0$. $F : (-\infty, \infty) \to \mathbb{R}$ is **absolutely continuous**

$$F(x) = \int_{-\infty}^{x} f(t)dt \text{ for some function } f.$$

Def : f is called the density of the distribution.

Properties : $\frac{d}{dx}F(x) = f(x)$ $\qquad$ $F(x) = \int_{-\infty}^{x} f(t)dt$

Cumulative Distribution Function (cdf)



From top to bottom:

- the cumulative distribution function of a **discrete** probability distribution
- **continuous** probability distribution,
- a distribution which has both a **continuous part and a discrete** part.

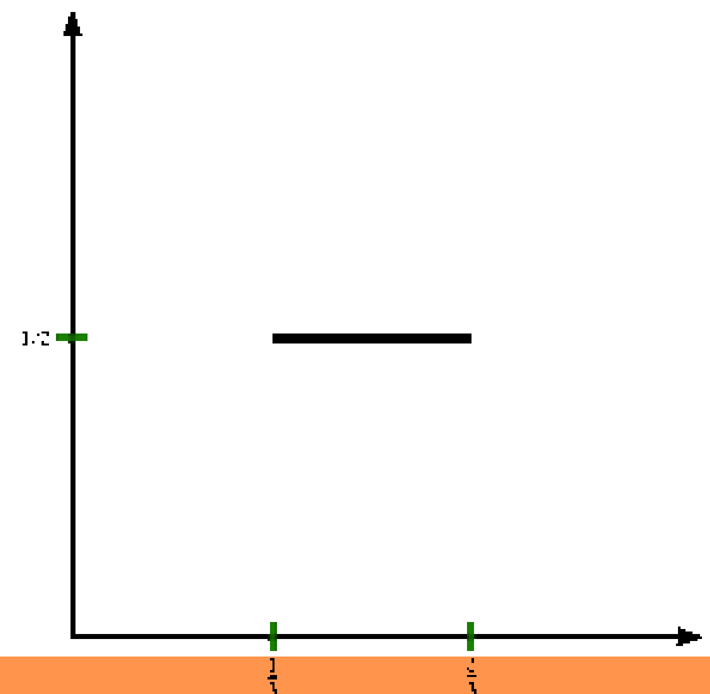
Cumulative Distribution Function (cdf)

If the CDF is **absolute continuous**, then the distribution has **density** function.

$$\frac{d}{dx}F(x) = f(x) \qquad\qquad F(x) = \int_{-\infty}^{x} f(t)dt$$

Why do we need **absolute** continuity?
Continuity of the CDF is not enough to have density function???

$$F(x) \neq \int_{-\infty}^{x} f(t)dt = 0$$



Cantor function: F continuous everywhere, has zero derivative (f=0) almost everywhere, F goes from 0 to 1 as x goes from 0 to 1, and takes on every value in between. $\Rightarrow$ there is **no density** for the Cantor function CDF.

Probability Density Function (pdf)

Pdf properties:

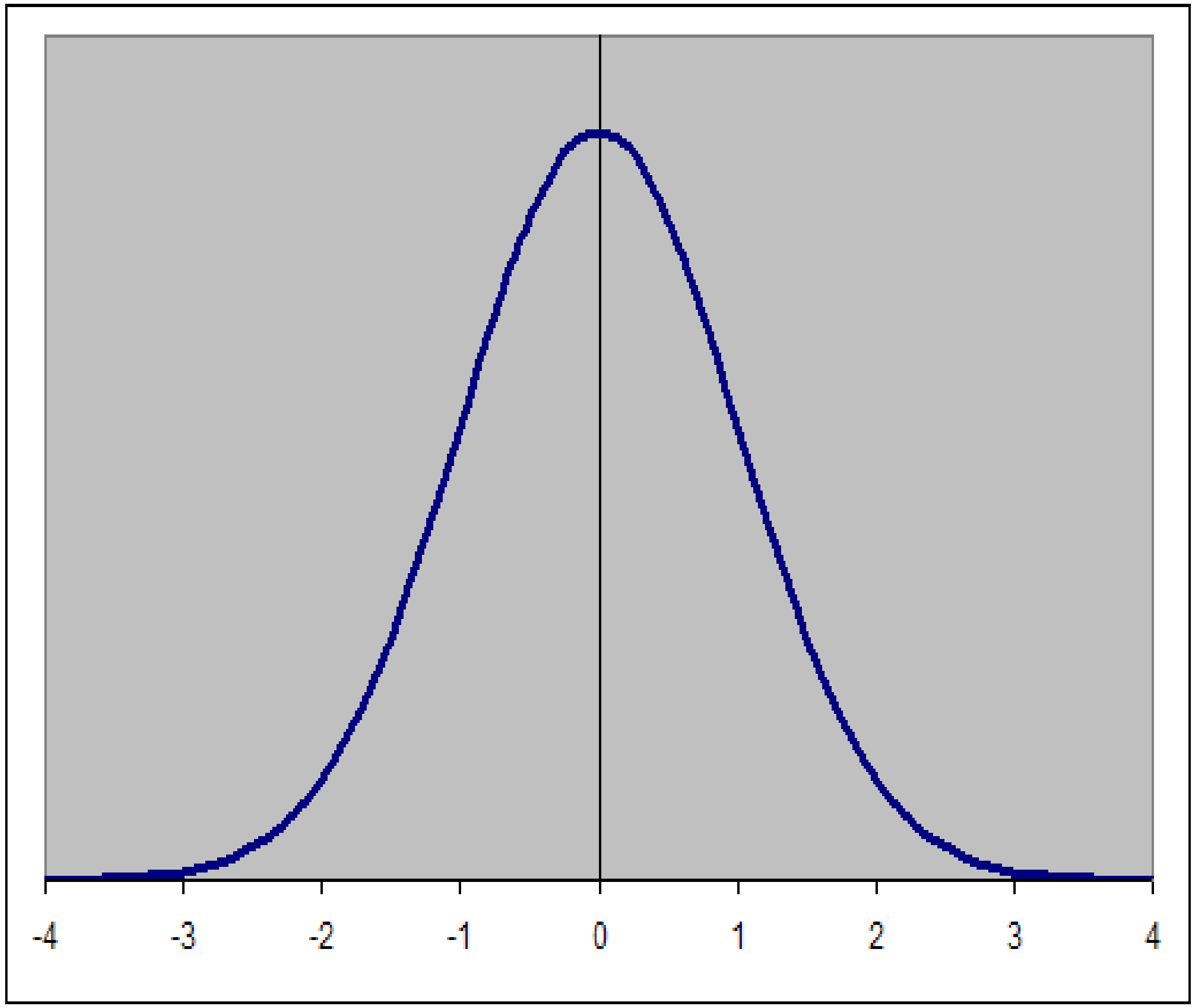
$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$f(x) = \frac{d}{dx}F(x)$$

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$$

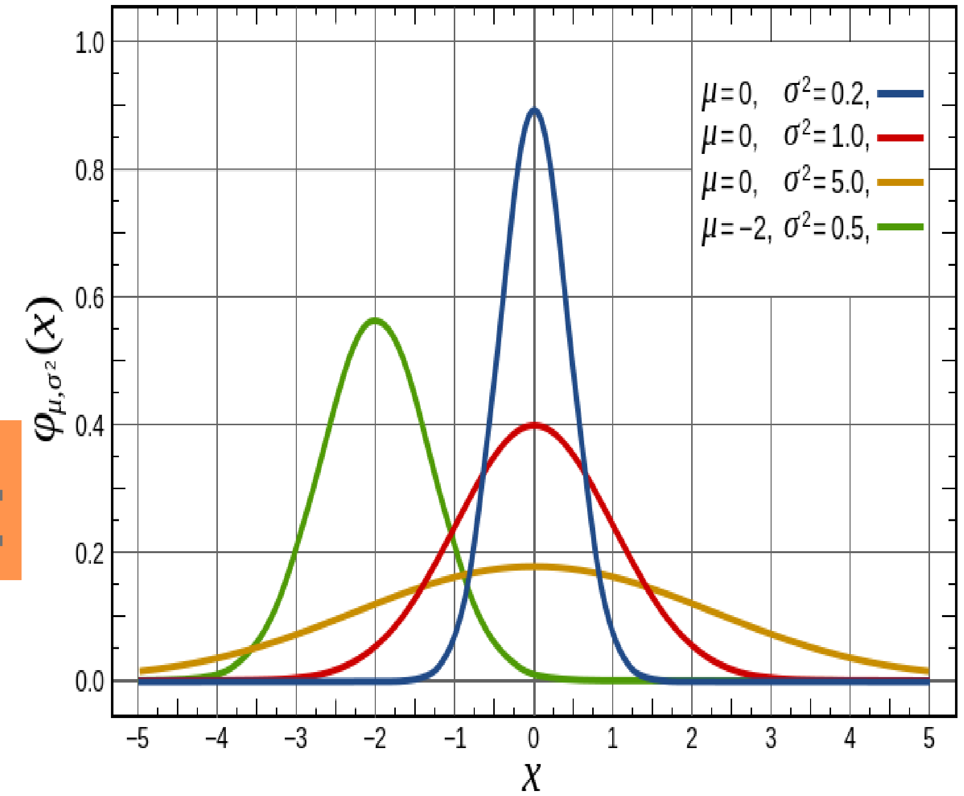


Intuitively, one can think of f(x)dx as being the probability of X falling within the infinitesimal interval [x, x + dx]. $P(x < X < x + dx) = f(x)dx$

Moments

Expectation: average value, mean, 1st moment:

$$E(X) = \begin{cases} \sum\limits_{i \in \Omega} x_i p(x_i) & \text{discrete} \\ \int\limits_{-\infty}^{\infty} x p(x) dx & \text{continuous} \end{cases}$$



Variance: the spread, 2nd moment:

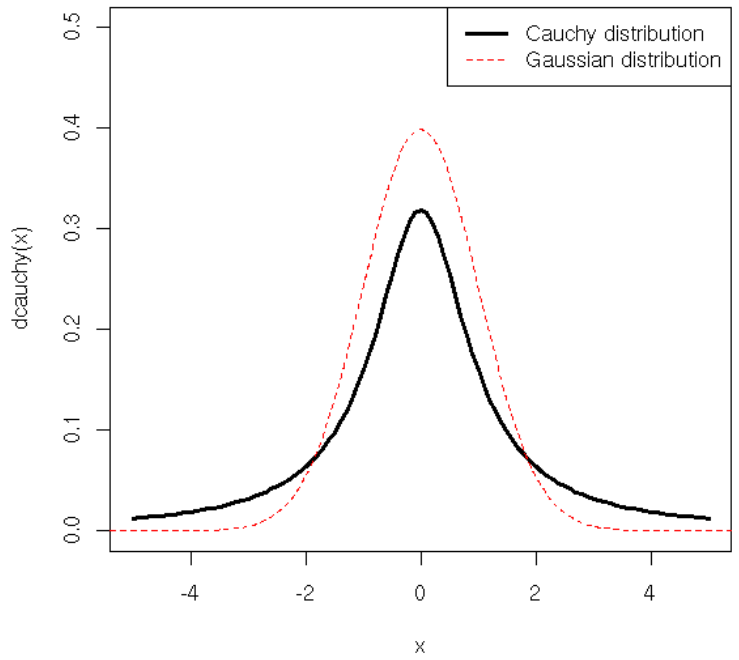
$$E(X) = \begin{cases} \sum\limits_{i \in \Omega} [x_i - E(X)]^2 p(x_i) & \text{discrete} \\ \int\limits_{-\infty}^{\infty} (x - E(x))^2 p(x) dx & \text{continuous} \end{cases}$$

Warning!

Moments may not always exist!

Cauchy distribution

$$p(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

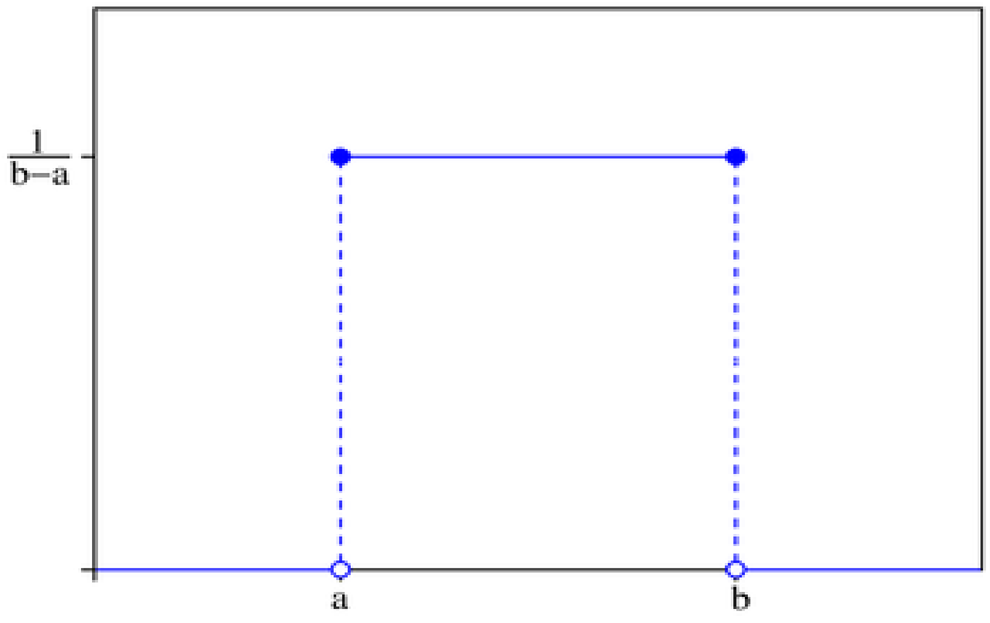


For the mean to exist the following integral would have to converge

$$\int_{-\infty}^{\infty} |x| p(x) dx = \int_{-\infty}^{\infty} |x| \frac{1}{\pi}\frac{1}{1+x^2} dx = 2\int_{0}^{\infty} x \frac{1}{\pi}\frac{1}{1+x^2} dx$$

$$\geq \frac{1}{\pi}\int_{1}^{\infty} \frac{2x}{1+x^2} dx \geq \frac{1}{\pi}\int_{1}^{\infty} \frac{1}{x} dx = \infty$$

Uniform Distribution



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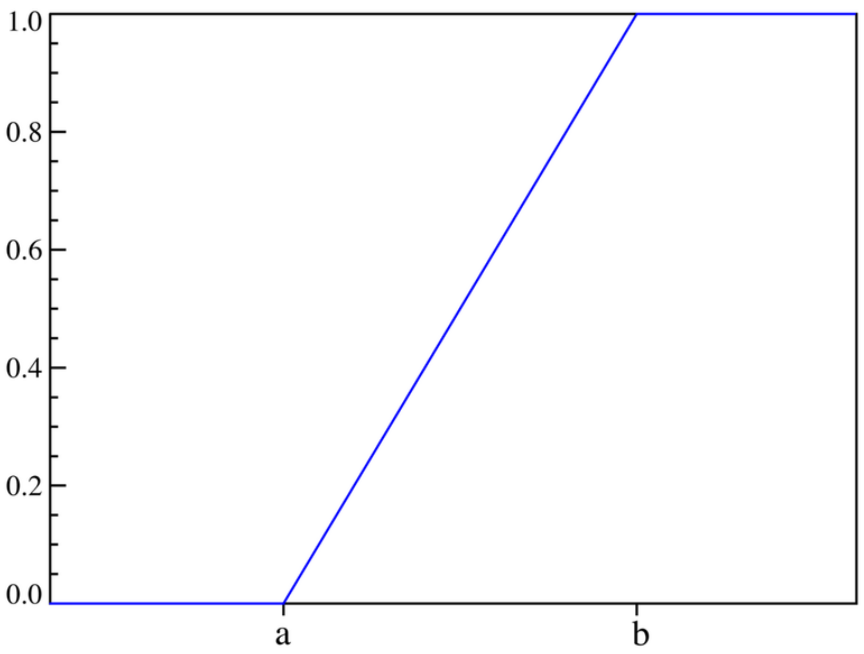
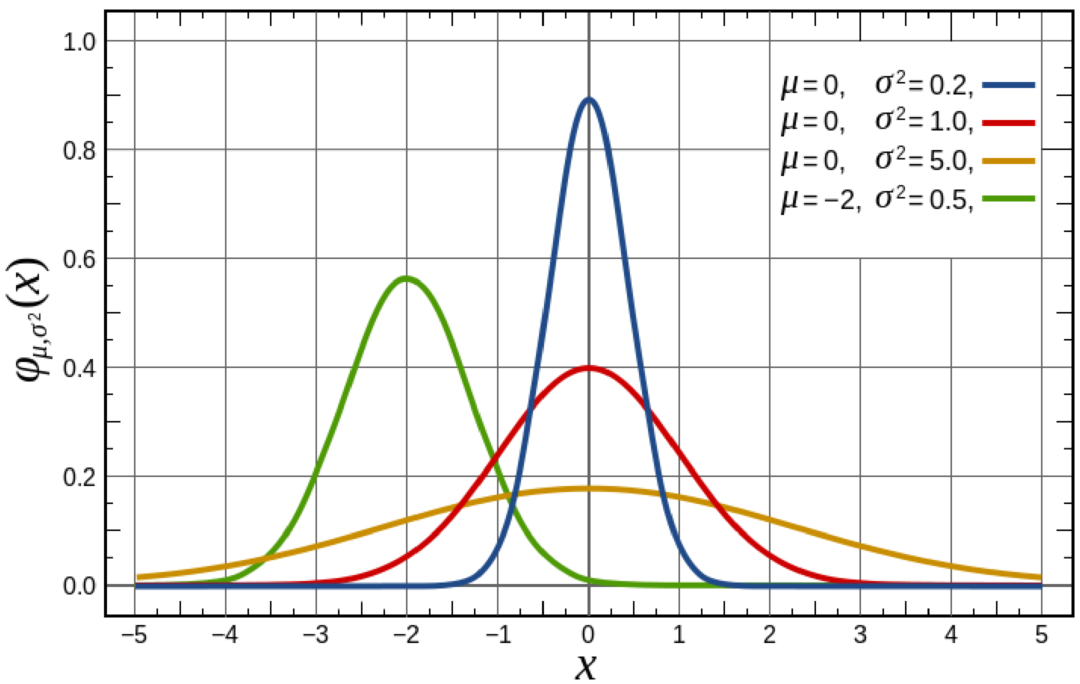


CDF

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & b < x \end{cases}$$

Normal (Gaussian) Distribution



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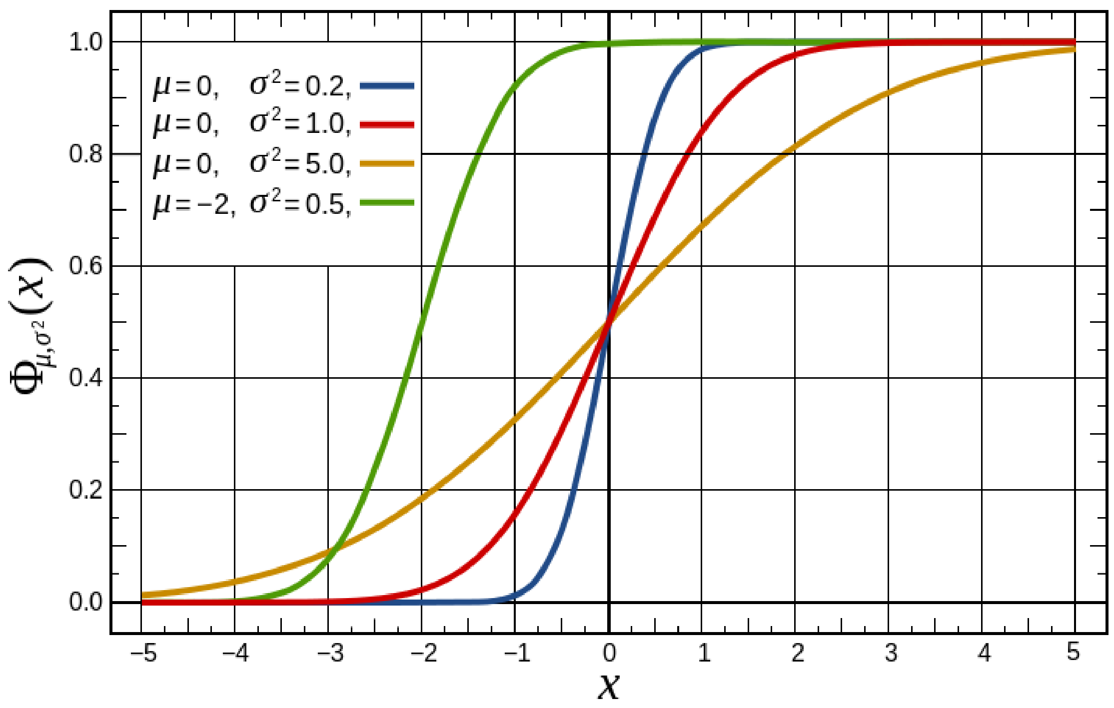


CDF

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$F(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)\right]$$

Multivariate (Joint) Distribution

We can generalize the above ideas from 1-dimension to any finite dimensions.

$P(a \leq X \leq b, c \leq Y \leq d) = ?$

$P(a_1 \leq X_1 \leq b_1, \ldots a_d \leq X_d \leq b_d) = ?$

Discrete distribution:

$P(\text{headache} \wedge \text{no flu}) = 7/80$

$P(\text{headache}) = 7/80 + 1/80$

$P(X = \text{headache}, Y = \text{flu}) = 1/80$

	Flu	No Flu
Headache	1/80	7/80
No Headache	1/80	71/80

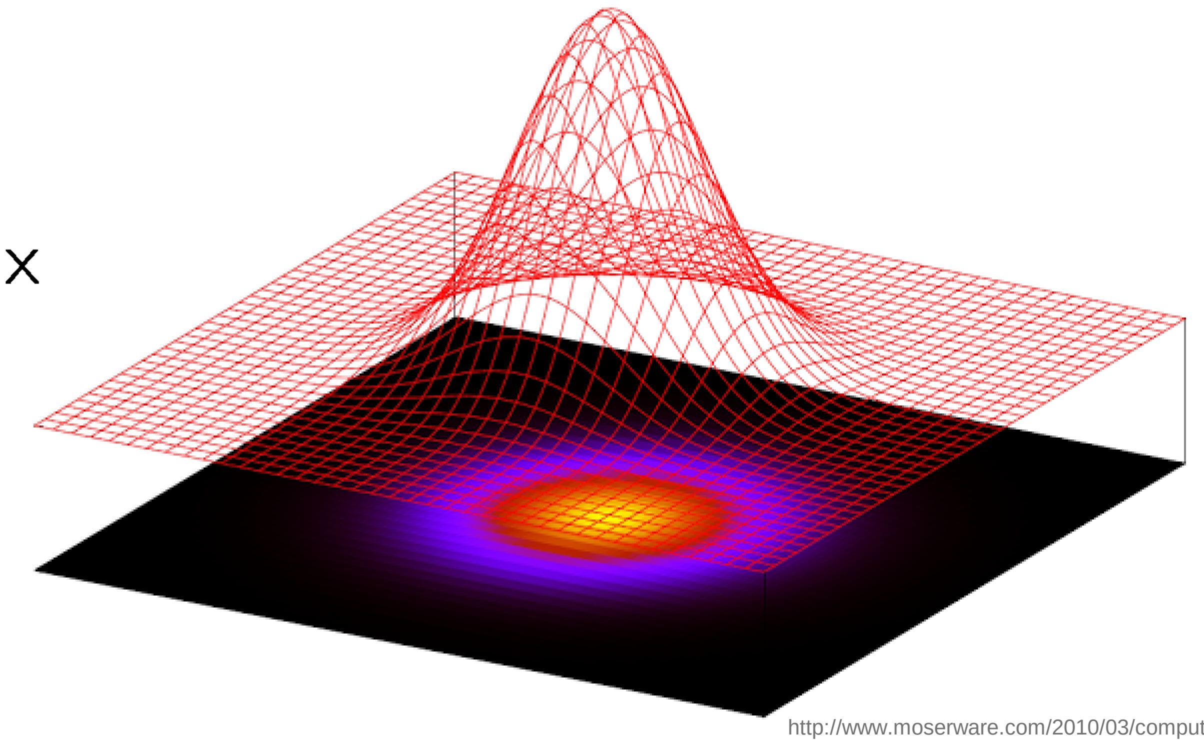
Multivariate Gaussian distribution

For $A \subset \mathbb{R}^d$, $P([X_1, \ldots, X_d] \in A) = \int_A f(x_1, \ldots, x_d) dx_1 \cdots dx_d$

$$F_X(z_1, \ldots, z_d) = \int_{-\infty}^{z_1} \cdots \int_{-\infty}^{z_d} f(x_1, \ldots, x_d) dx_1 \cdots dx_d$$ Multivariate CDF

$\mu \in \mathbb{R}^d$: mean vector

$\Sigma \in \mathbb{R}^{d \times d}$: covariance matrix



http://www.moserware.com/2010/03/computing-your-skill.htm

$$f_X(x_1, \ldots, x_d) = \frac{1}{\sqrt{|2\pi\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)\right)$$

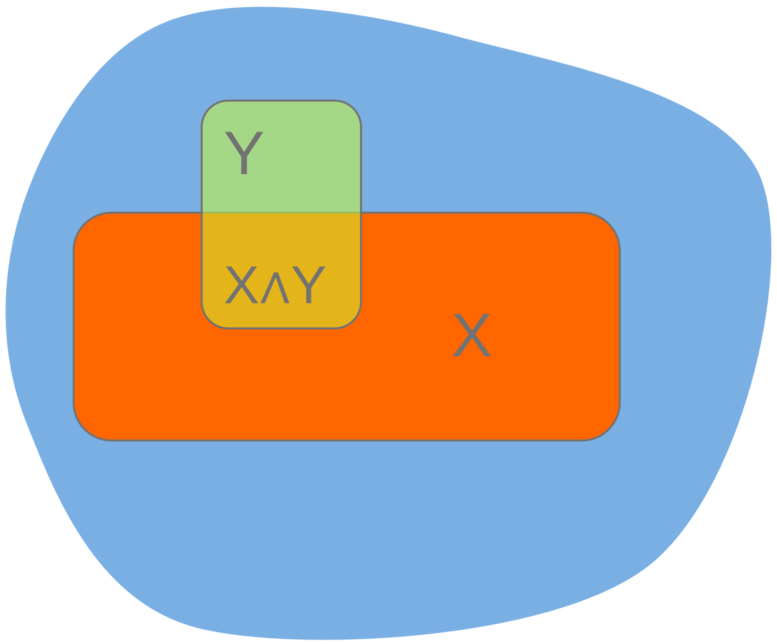
Conditional Probability

P(X|Y) = Fraction of worlds in which X event is true given Y event is true.

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}$$

$$P(\text{flu}|\text{headache}) = \frac{P(\text{flu, headache})}{P(\text{headache})} = \frac{1/80}{1/80 + 7/80}$$

	Flu	No Flu
Headache	1/80	7/80
No Headache	1/80	71/80



Independence

Independent random variables:

$$P(X,Y) = P(X)P(Y)$$

$$P(X|Y) = P(X)$$

Y and X don't contain information about each other.
Observing Y doesn't help predicting X.
Observing X doesn't help predicting Y.

Examples:

Independent: Winning on roulette this week and next week.
Dependent: Russian roulette

Conditionally Independent

Conditionally independent:

$$P(X,Y|Z) = P(X|Z)P(Y|Z)$$

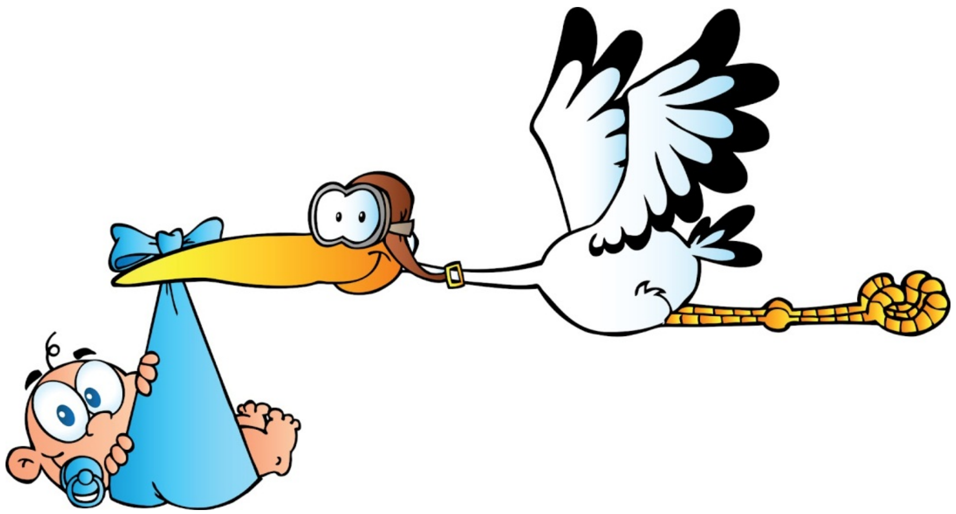
Knowing Z makes X and Y independent

Examples:

Dependent: show size and reading skills
Conditionally independent: show size and reading skills given age

Storks deliver babies:

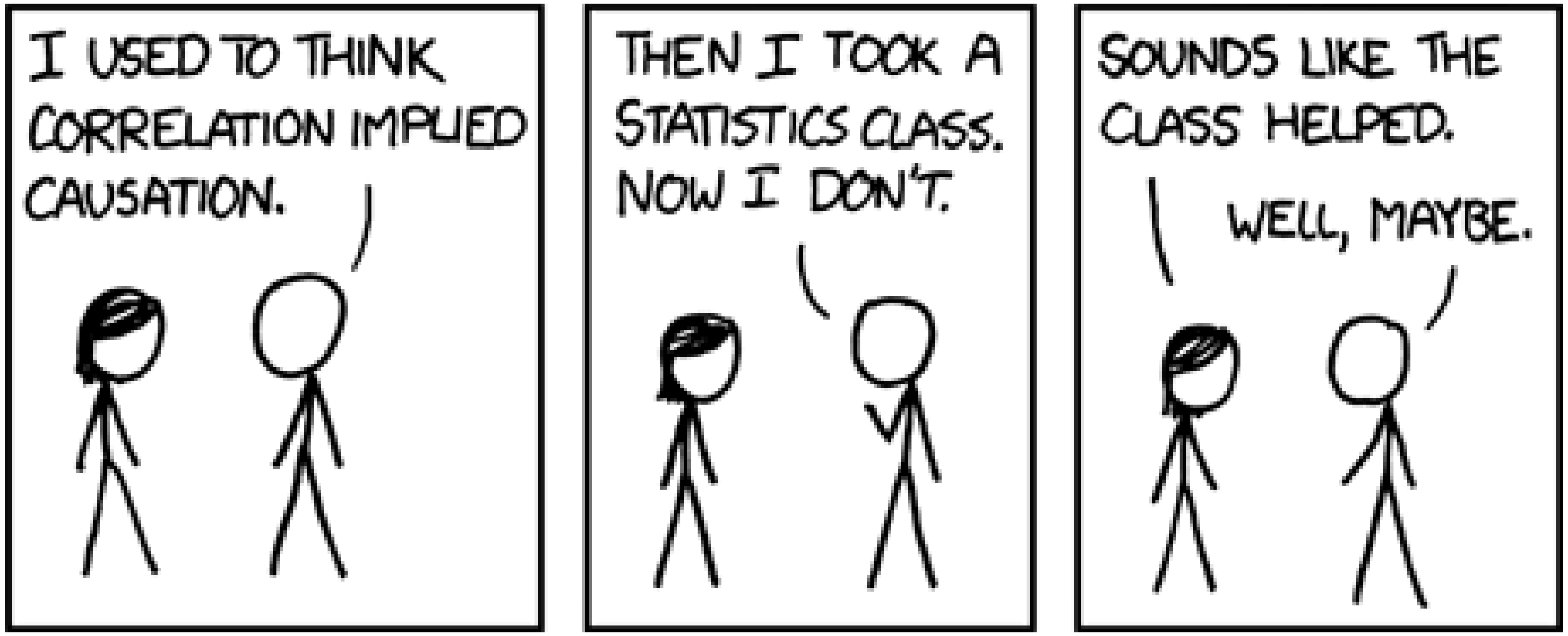
Highly statistically significant correlation exists between stork populations and human birth rates across Europe



Conditionally Independent

London taxi drivers: A survey has pointed out a positive and significant correlation between the number of accidents and wearing coats. They concluded that coats could hinder movements of drivers and be the cause of accidents. A new law was prepared to prohibit drivers from wearing coats when driving.

Finally another study pointed out that people wear coats when it rains…



xkcd.com

Conditional Independence

Formally: X is **conditionally independent** of Y given Z:

$$P(X, Y | Z) = P(X | Z) P(Y | Z)$$

$$P(\text{Accidents}, \text{Coats} \mid \text{Rain}) = P(\text{Accidents} \mid \text{Rain}) P(\text{Coats} \mid \text{Rain})$$

Equivalent to:

$$(\forall x, y, z) P(X = x | Y = y, Z = z) = P(X = x | Z = z)$$

Bayes Rule

Chain Rule & Bayes Rule

Chain rule:

$$P(X,Y) = P(X|Y)P(Y) = P(Y|X)P(X)$$

Bayes rule: $$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

Bayes rule is important for reverse conditioning.

AIDS test (Bayes rule)

Data

- Approximately 0.1% are infected
- Test detects all infections
- Test reports positive for 1% healthy people

Probability of having AIDS if test is positive:

$$P(a=1|t=1) = \frac{P(t=1|a=1)P(a=1)}{P(t=1)}$$

$$= \frac{P(t=1|a=1)P(a=1)}{P(t=1|a=1)P(a=1) + P(t=1|a=0)P(a=0)}$$

$$= \frac{1 \cdot 0.001}{1 \cdot 0.001 + 0.01 \cdot 0.999} = 0.091$$

Only 9%!...

Improving the diagnosis

Use a follow-up test!

- Test 2 reports positive for 90% infections
- Test 2 reports positive for 5% healthy people

$$P(a=0|t_1=1,t_2=1) = \frac{P(t_1=1,t_2=1|a=0)P(a=0)}{P(t_1=1,t_2=1|a=1)P(a=1)+P(t_1=1,t_2=1|a=0)P(a=0)}$$

$$= \frac{0.01 \cdot 0.05 \cdot 0.999}{1 \cdot 0.9 \cdot 0.001 + 0.01 \cdot 0.05 \cdot 0.999} = 0.357$$

$$P(a=1|t_1=1,t_2=1) = 0.643$$

Why can't we use Test 1 twice?

Outcomes are **not** independent but tests 1 and 2 are **conditionally independent** $p(t_1, t_2|a) = p(t_1|a) \cdot p(t_2|a)$

Application:
Document Classification, Spam filtering

Data for spam filtering

- date
- time
- recipient path
- IP number
- sender
- encoding
- many more features

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Naïve Bayes Assumption

Naïve Bayes assumption: Features X_1 and X_2 are conditionally independent given the class label Y:

$$P(X_1, X_2|Y) = P(X_1|Y)P(X_2|Y)$$

More generally:
$$P(X_1 ... X_d|Y) = \prod_{i=1}^{d} P(X_i|Y)$$

How many parameters to estimate?

(X is composed of d binary features, e.g. presence of "earn"
Y has K possible class labels)

$\boxed{(2^d\text{-}1)K \text{ vs } (2\text{-}1)dK}$

Naïve Bayes Classifier

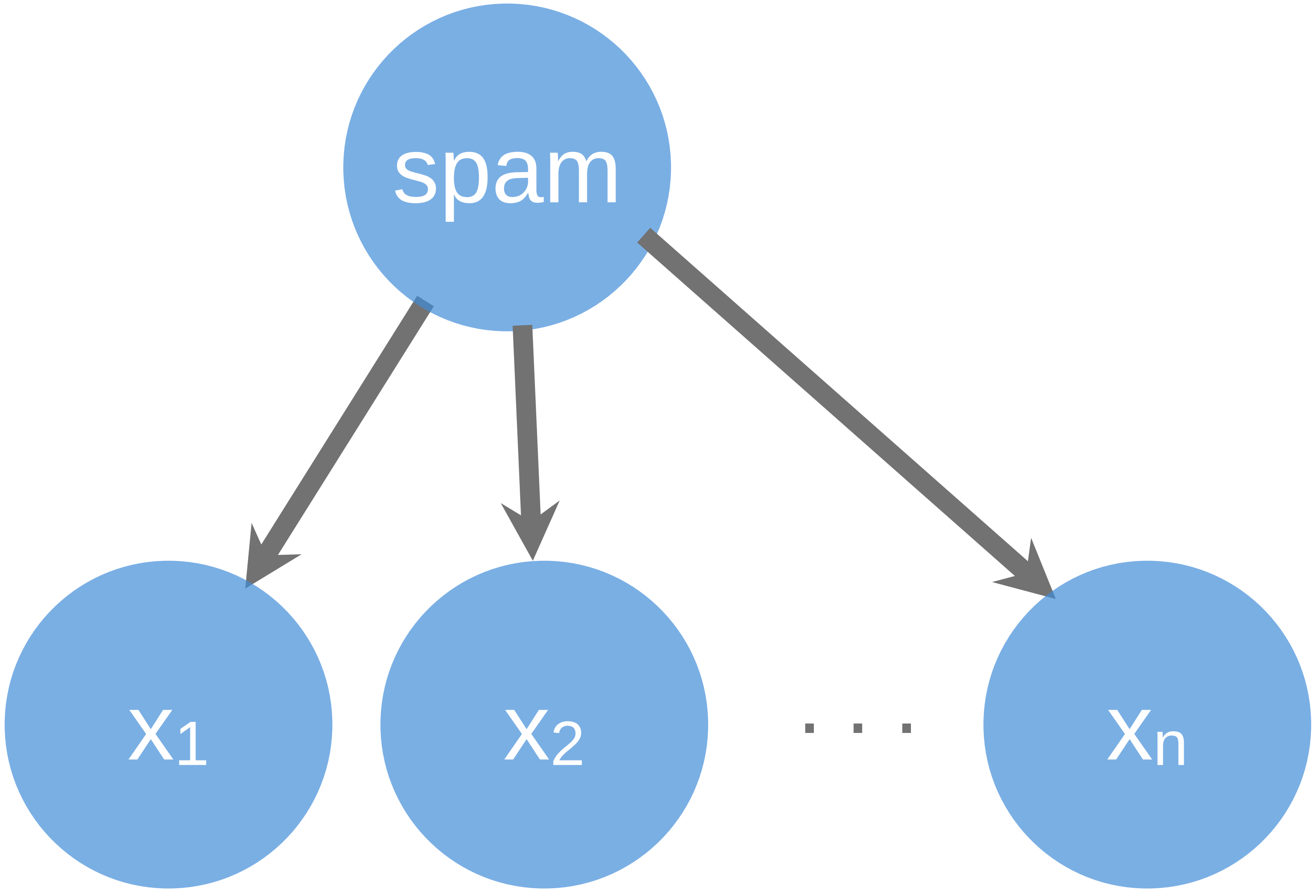
Given:

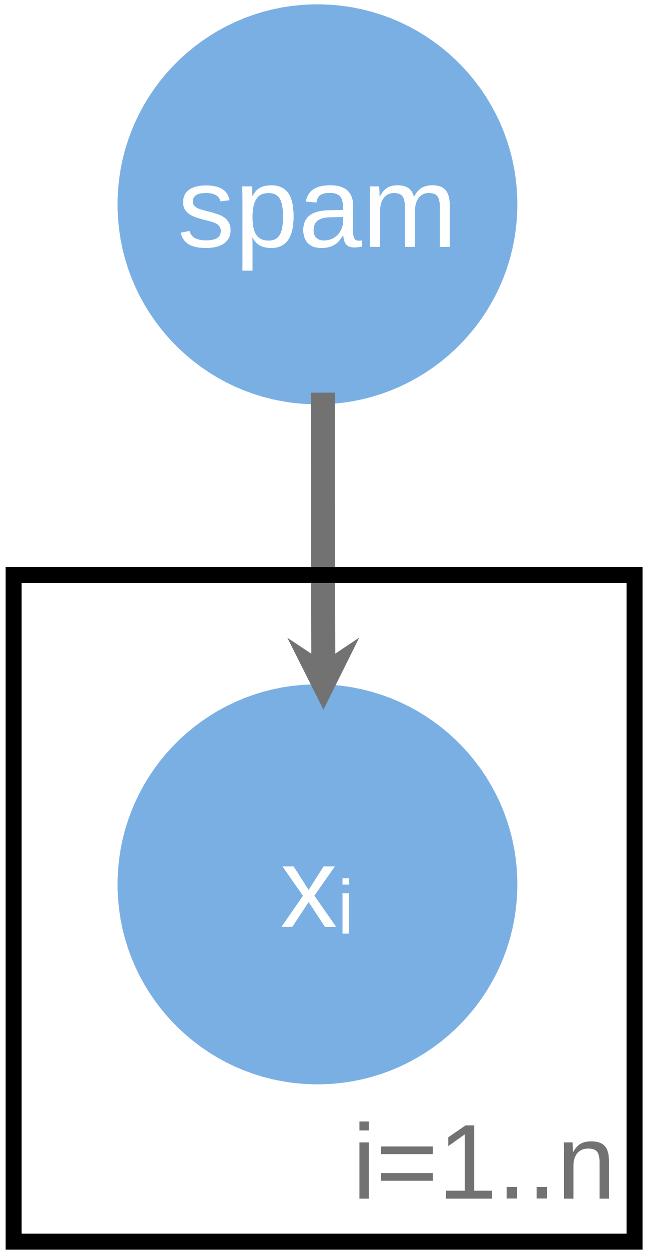
- Class prior $P(Y)$
- d conditionally independent features $X_1,\ldots X_d$ given the class label Y
- For each X_i, we have the conditional likelihood $P(X_i|Y)$

Decision rule:

$$
\begin{aligned}
f_{NB}(\mathbf{x}) &= \arg\max_y \; P(x_1, \ldots, x_d \mid y) P(y) \\
&= \arg\max_y \prod_{i=1}^{d} P(x_i|y) P(y)
\end{aligned}
$$

A Graphical Model





$$P(X_1, \ldots, X_d \mid \text{spam}) = \prod_{i=1}^{d} P(X_i|\text{spam})$$

Naïve Bayes Algorithm for discrete features

Training Data: $\{(X^{(j)}, Y^{(j)})\}_{j=1}^{n}$ $X^{(j)} = (X_1^{(j)}, \ldots, X_d^{(j)})$

n d dimensional features + class labels

$$f_{NB}(\mathbf{x}) = \arg\max_y \prod_{i=1}^{d} P(x_i|y)P(y)$$

We need to estimate these probabilities!

Estimate them with Relative Frequencies!

For Class Prior $\widehat{P}(y) = \frac{\{\#j : Y^{(j)} = y\}}{n}$

For Likelihood $\frac{\widehat{P}(x_i, y)}{\widehat{P}(y)} = \frac{\{\#j : X_i^{(j)} = x_i, Y^{(j)} = y\}/n}{\{\#j : Y^{(j)} = y\}/n}$

NB Prediction for test data:

$X = (x_1, \ldots, x_d)$

$$\boxed{Y = \arg\max_y \widehat{P}(y) \prod_{i=1}^{d} \frac{\widehat{P}(x_i, y)}{\widehat{P}(y)}}$$

Subtlety: Insufficient training data

What if you never see a training instance where $X_1 = a$ when $Y = b$?

For example,

there is no X_1=‘Earn’ when Y=‘SpamEmail’ in our dataset.

$$\Rightarrow P(X_1 = a, Y = b) = 0 \Rightarrow P(X_1 = a|Y = b) = 0$$

$$\boxed{\Rightarrow P(X_1 = a, X_2 \ldots X_n|Y) = P(X_1 = a|Y) \prod_{i=2}^{d} P(X_i|Y) = 0}$$

Thus, no matter what the values $X_2, \ldots, X_d$ take:

$$P(Y = b \mid X_1 = a, X_2, \ldots, X_d) = 0$$

What now???

Parameter estimation: MLE, MAP

Estimating Probabilities

Flipping a Coin

I have a coin, if I flip it, what's the probability it will fall with the head up?

Let us flip it a few times to estimate the probability:

The estimated probability is: **3/5** "Frequency of heads"

Why???... and How good is this estimation???

MLE for Bernoulli distribution

Data, D =

$$D = \{X_i\}_{i=1}^n, \; X_i \in \{\mathrm{H}, \mathrm{T}\}$$

P(Heads) = θ*, P(Tails) = 1-*θ

Flips are **i.i.d.**:
- **Independent** events
  - **Identically distributed** according to Bernoulli distribution

MLE: Choose θ that maximizes the probability of observed data

Maximum Likelihood Estimation

MLE: Choose θ that maximizes the probability of observed data

$$\widehat{\theta}_{MLE} = \arg\max_{\theta} P(D \mid \theta)$$

$$= \arg\max_{\theta} \prod_{i=1}^{n} P(X_i|\theta)$$

Independent draws

$$= \arg\max_{\theta} \prod_{i:X_i=H} \theta \prod_{i:X_i=T} (1-\theta)$$

Identically distributed

$$= \arg\max_{\theta} \underbrace{\theta^{\alpha_H}(1-\theta)^{\alpha_T}}_{J(\theta)}$$

Maximum Likelihood Estimation

MLE: Choose θ that maximizes the probability of observed data

$$\widehat{\theta}_{MLE} = \arg\max_{\theta} \; P(D \mid \theta)$$

$$= \arg\max_{\theta} \; \underbrace{\theta^{\alpha_H}(1-\theta)^{\alpha_T}}_{J(\theta)}$$

$$\frac{\partial J(\theta)}{\partial \theta} = \alpha_H \theta^{\alpha_H - 1}(1-\theta)^{\alpha_T} - \alpha_T \theta^{\alpha_H}(1-\theta)^{\alpha_T - 1}\Big|_{\theta=\hat{\theta}_{\mathrm{MLE}}} = 0$$

$$\alpha_H(1-\theta) - \alpha_T\theta\Big|_{\theta=\hat{\theta}_{\mathrm{MLE}}} = 0$$

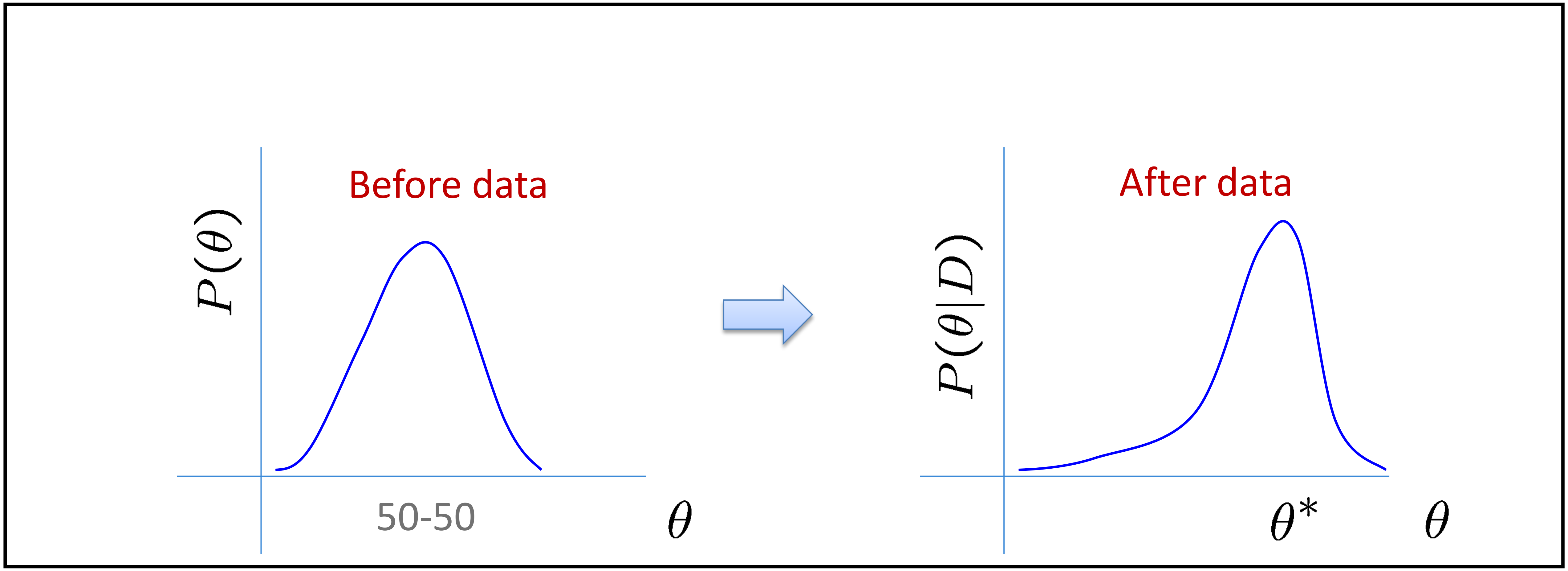
$$\boxed{\widehat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}}$$

What about prior knowledge?

We know the coin is “close” to 50-50. What can we do now?

The Bayesian way…

Rather than estimating a single θ, we obtain a distribution over possible values of θ



Bayesian Learning

- Use Bayes rule:

$$P(\theta \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid \theta)P(\theta)}{P(\mathcal{D})}$$

- Or equivalently:

$$P(\theta \mid \mathcal{D}) \propto P(\mathcal{D} \mid \theta)P(\theta)$$

posterior likelihood prior

Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London*, **53:370-418**

MAP estimation for Binomial distribution

Coin flip problem

Likelihood is Binomial $$P(\mathcal{D} \mid \theta) = \binom{n}{\alpha_H} \theta^{\alpha_H} (1-\theta)^{\alpha_T}$$

If the prior is Beta distribution,

$$P(\theta) = \frac{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$

$\Rightarrow$ posterior is Beta distribution

$$P(\theta|D) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$

P(θ) and P(θ| D) have the same form! [Conjugate prior]

$$\widehat{\theta}_{MAP} = \arg\max_{\theta} P(\theta \mid D) = \arg\max_{\theta} P(D \mid \theta) P(\theta) = \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2}$$

MLE vs. MAP

- Maximum Likelihood estimation (MLE)

  Choose value that maximizes the probability of observed data

$$\widehat{\theta}_{MLE} = \arg\max_{\theta} P(D|\theta)$$

- Maximum *a posteriori* (MAP) estimation

  Choose value that is most probable given observed data and prior belief

$$\begin{aligned}\widehat{\theta}_{MAP} &= \arg\max_{\theta} P(\theta|D) \\ &= \arg\max_{\theta} P(D|\theta)P(\theta)\end{aligned}$$

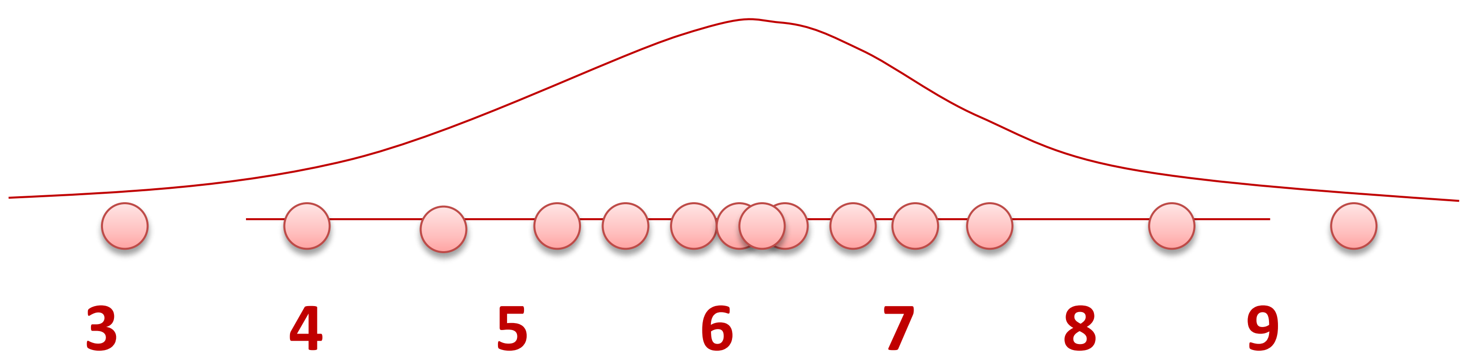
When is MAP same as MLE?

Bayesians vs.Frequentists

You are no good when sample is small

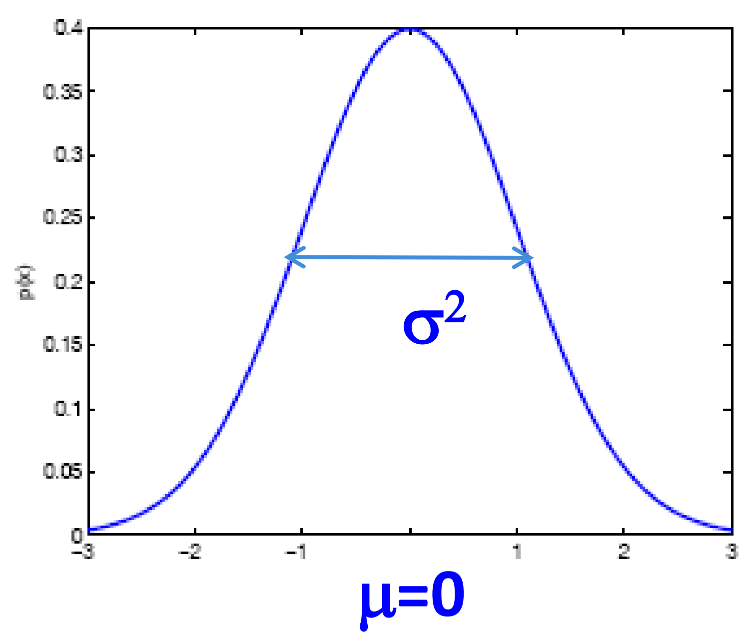
You give a different answer for different priors

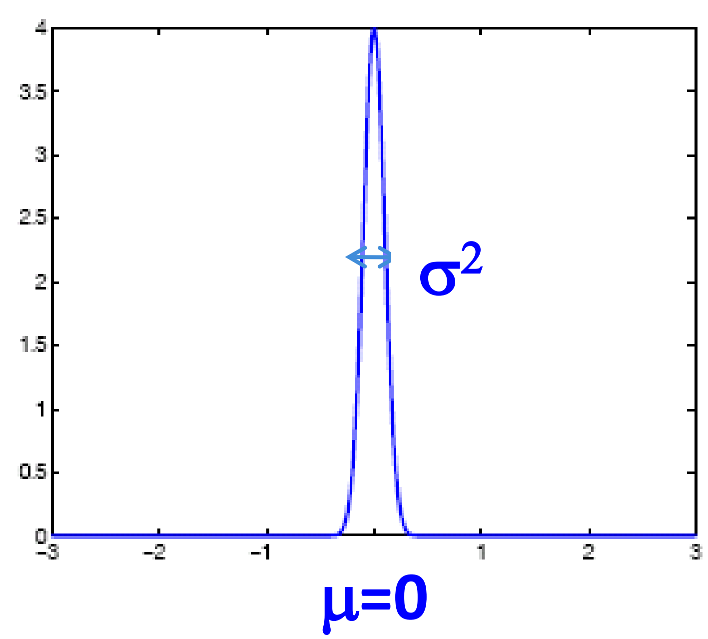
What about continuous features?



Let us try Gaussians...

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) = \mathcal{N}_x(\mu, \sigma)$$





MLE for Gaussian mean and variance

Choose $\theta = (\mu, \sigma^2)$ that maximizes the probability of observed data

$$\widehat{\theta}_{MLE} = \arg\max_{\theta} P(D \mid \theta)$$

$$= \arg\max_{\theta} \prod_{i=1}^{n} P(X_i|\theta)$$

Independent draws

$$= \arg\max_{\theta} \prod_{i=1}^{n} \frac{1}{2\sigma^2} e^{-(X_i-\mu)^2/2\sigma^2}$$

Identically distributed

$$= \arg \max_{\theta=(\mu,\sigma^2)} \underbrace{\frac{1}{2\sigma^2} e^{-\sum_{i=1}^{n}(X_i-\mu)^2/2\sigma^2}}_{J(\theta)}$$

MLE for Gaussian mean and variance

$$\hat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

Note: MLE for the variance of a Gaussian is **biased**

[Expected result of estimation is **not** the true parameter!]

Unbiased variance estimator: $\hat{\sigma}^2_{unbiased} = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \hat{\mu})^2$

Case Study: Text Classification

Case Study: Text Classification

- Classify e-mails
  - Y = {Spam,NotSpam}
- Classify news articles
  - Y = {what is the topic of the article?

What about the features **X**?

The text!

X_i represents i^{th} word in document

Article from rec.sport.hockey

```
Path: cantaloupe.srv.cs.cmu.edu!das-news.harvard.e
From: xxx@yyy.zzz.edu (John Doe)
Subject: Re: This year's biggest and worst (opinic
Date: 5 Apr 93 09:53:39 GMT

I can only comment on the Kings, but the most
obvious candidate for pleasant surprise is Alex
Zhitnik. He came highly touted as a defensive
defenseman, but he's clearly much more than that.
Great skater and hard shot (though wish he were
more accurate). In fact, he pretty much allowed
the Kings to trade away that huge defensive
liability Paul Coffey. Kelly Hrudey is only the
biggest disappointment if you thought he was any
good to begin with. But, at best, he's only a
mediocre goaltender. A better choice would be
Tomas Sandstrom, though not through any fault of
his own, but because some thugs in Toronto decided
```

NB for Text Classification

P(**X**|Y) is huge!!!

- Article at least 1000 words, **X**=$\{X_1,\ldots,X_{1000}\}$
- X_i represents i^{th} word in document, i.e., the domain of X_i is entire vocabulary, e.g., Webster Dictionary (or more). $X_i \in \{1,\ldots,50000\} \Rightarrow K1000^{50000}$ parameters….

NB assumption helps a lot!!!

- $P(X_i=x_i|Y=y)$ is the probability of observing word x_i at the i^{th} position in a document on topic y $\Rightarrow$ 1000K50000 parameters

$$h_{NB}(\mathbf{x}) = \arg\max_y P(y) \prod_{i=1}^{LengthDoc} P(x_i|y)$$

Bag of words model

Typical additional assumption – **Position in document doesn't matter**: $P(X_i=x_i|Y=y) = P(X_k=x_i|Y=y)$

- "Bag of words" model – order of words on the page ignored
- Sounds really silly, but often works very well! $\Rightarrow$ K50000 parameters

$$\prod_{i=1}^{LengthDoc} P(x_i|y) = \prod_{w=1}^{W} P(w|y)^{count_w}$$

When the lecture is over, remember to wake up the person sitting next to you in the lecture room.

Bag of words model

Typical additional assumption – **Position in document doesn't matter**: $P(X_i=x_i|Y=y) = P(X_k=x_i|Y=y)$

- "Bag of words" model – order of words on the page ignored
- Sounds really silly, but often works very well!

$$\prod_{i=1}^{LengthDoc} P(x_i|y) = \prod_{w=1}^{W} P(w|y)^{count_w}$$

in is lecture lecture next over person remember room sitting the the the to to up wake when you

Bag of words approach

the world of
TOTAL

▶ All About The Company
Global Activities
Corporate Structure
TOTAL's Story
Upstream Strategy
Downstream Strategy
Chemicals Strategy
TOTAL Foundation
Homepage

all about the **company**

Our energy exploration, production, and distribution operations span the globe, with activities in more than 100 countries.

At TOTAL, we draw our greatest strength from our fast-growing oil and gas reserves. Our strategic emphasis on natural gas provides a strong position in a rapidly expanding market.

Our expanding refining and marketing operations in Asia and the Mediterranean Rim complement already solid positions in Europe, Africa, and the U.S.

Our growing specialty chemicals sector adds balance and profit to the core energy business.

aardvark	0
about	2
all	2
Africa	1
apple	0
anxious	0
...	
gas	1
...	
oil	1
…	
Zaire	0

Twenty news groups results

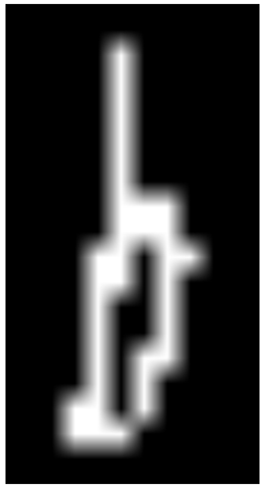
Given 1000 training documents from each group
Learn to classify new documents according to which newsgroup it came from

comp.graphics	misc.forsale
comp.os.ms-windows.misc	rec.autos
comp.sys.ibm.pc.hardware	rec.motorcycles
comp.sys.mac.hardware	rec.sport.baseball
comp.windows.x	rec.sport.hockey
alt.atheism	sci.space
soc.religion.christian	sci.crypt
talk.religion.misc	sci.electronics
talk.politics.mideast	sci.med
talk.politics.misc	
talk.politics.guns	

Naïve Bayes: 89% accuracy

What if features are continuous?

Eg., character recognition: X_i is intensity at i^{th} pixel



Gaussian Naïve Bayes (GNB):

$$P(X_i = x \mid Y = y_k) = \frac{1}{\sigma_{ik}\sqrt{2\pi}} \; e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}$$

Different mean and variance for each class k and each pixel i.

Sometimes assume variance

- is independent of Y (i.e., σ_i),
- or independent of X_i (i.e., σ_k)
- or both (i.e., σ)

Example: GNB for classifying mental states

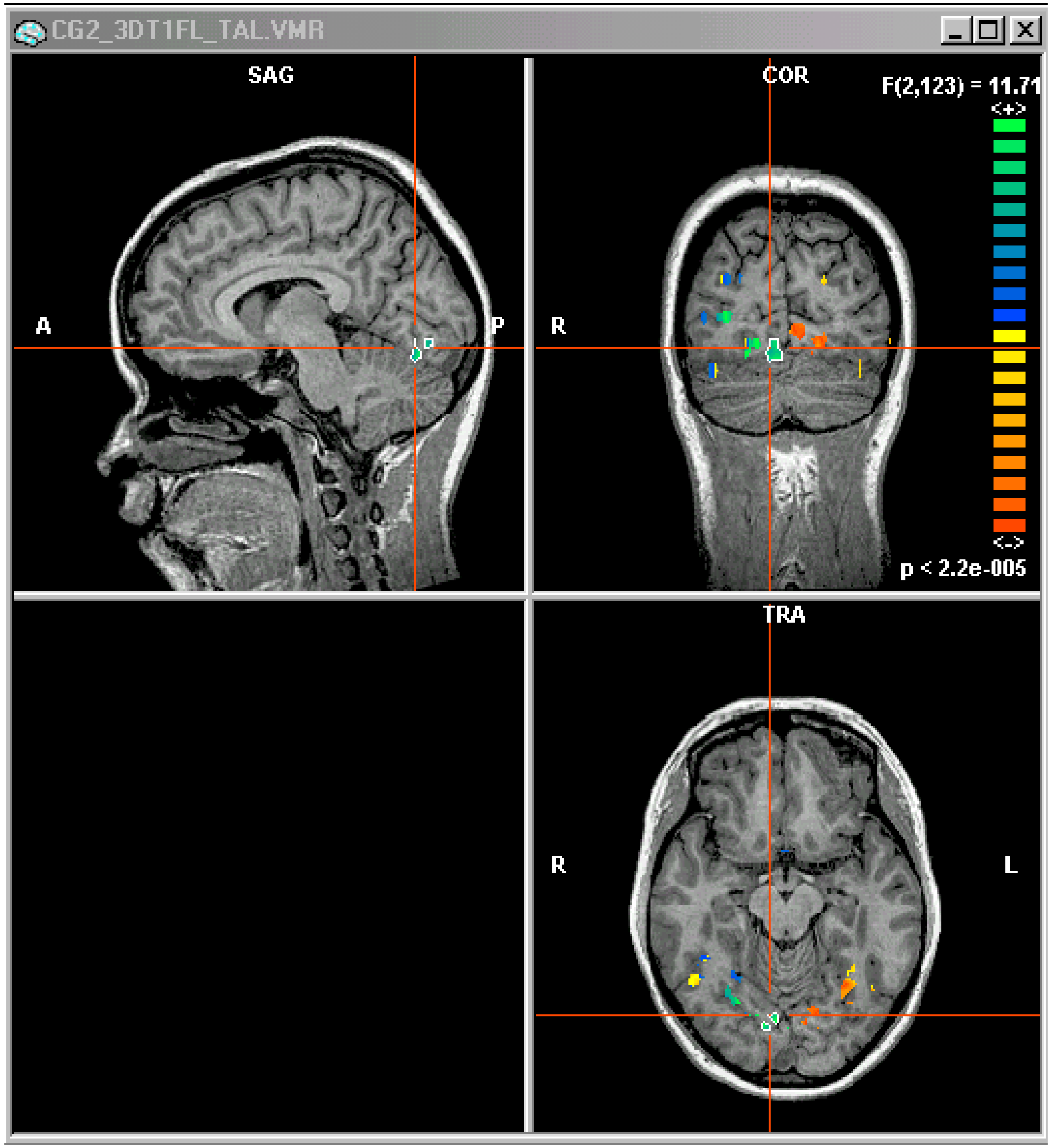
~1 mm resolution

~2 images per sec.

15,000 voxels/image

non-invasive, safe

measures Blood Oxygen Level Dependent (BOLD) response



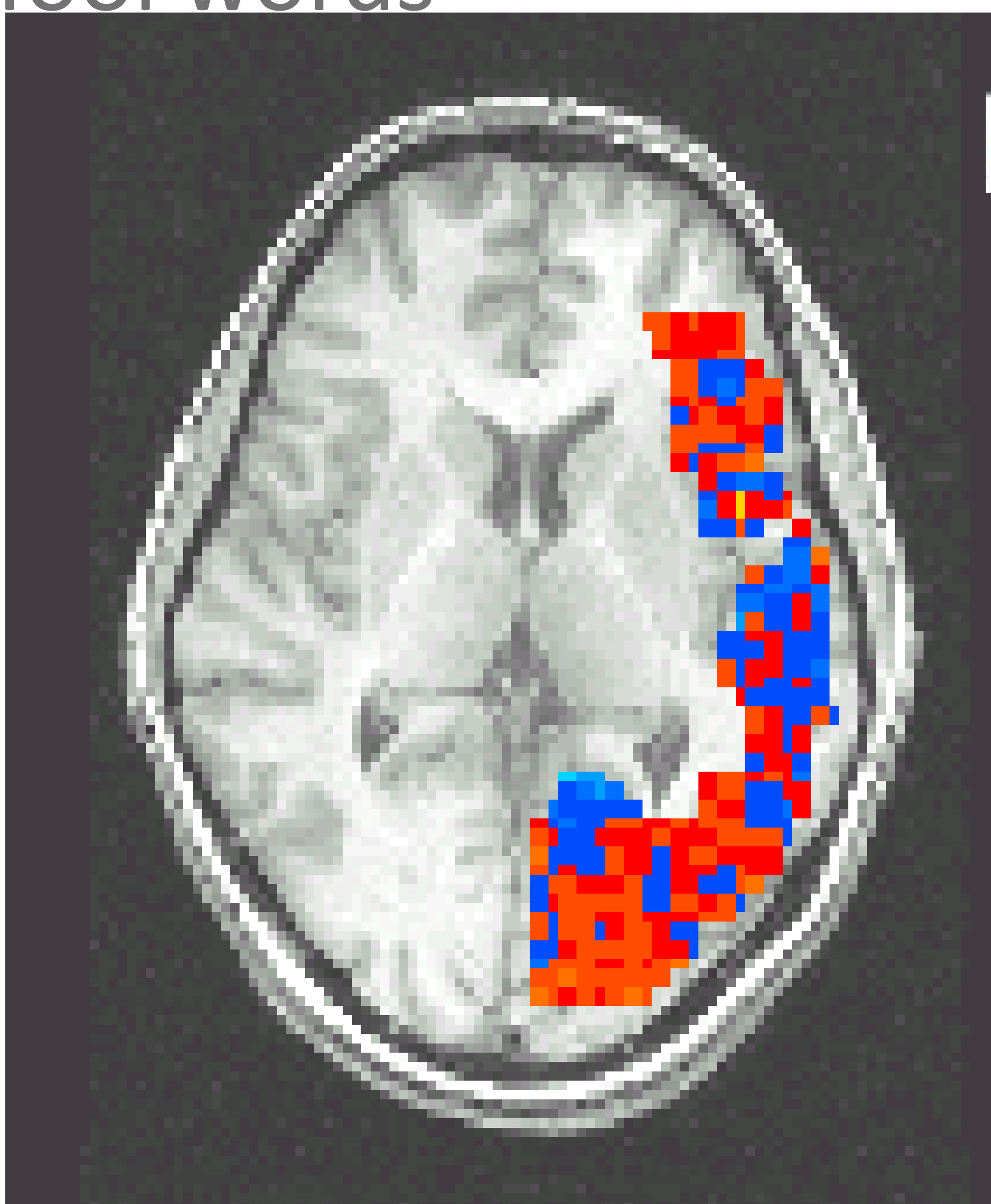
[Mitchell et al.]

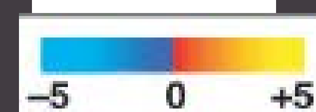
Learned Naïve Bayes Models – Means for P(BrainActivity | WordCategory)

Pairwise classification accuracy:
78-99%, 12 participants

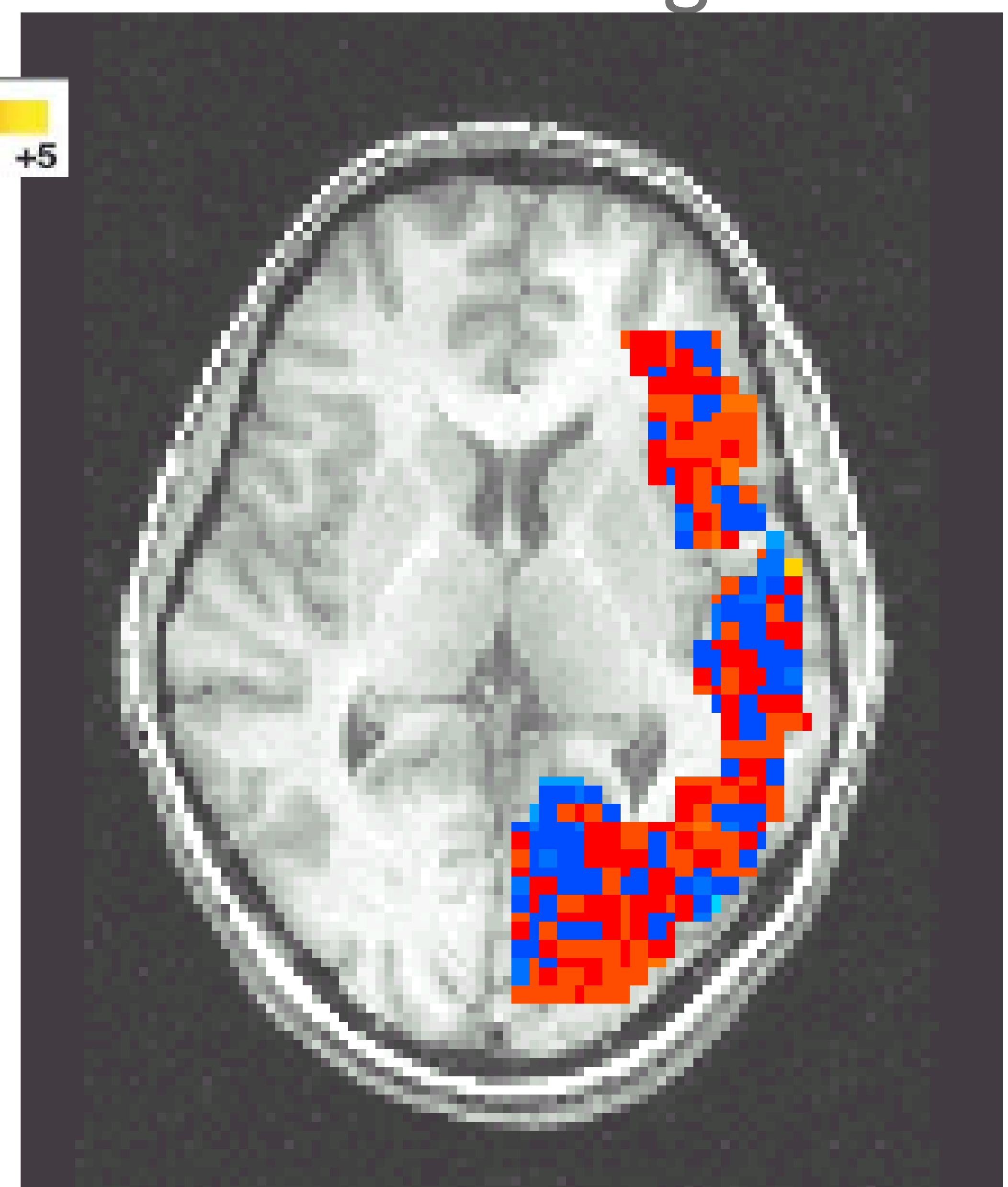
[Mitchell et al.]

Tool words





Building words



What you should know…

Naïve Bayes classifier

- What's the assumption
- Why we use it
- How do we learn it
- Why is Bayesian (MAP) estimation important

Text classification

- Bag of words model

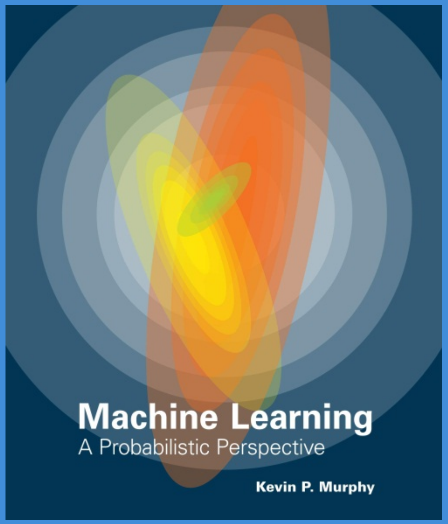
Gaussian NB

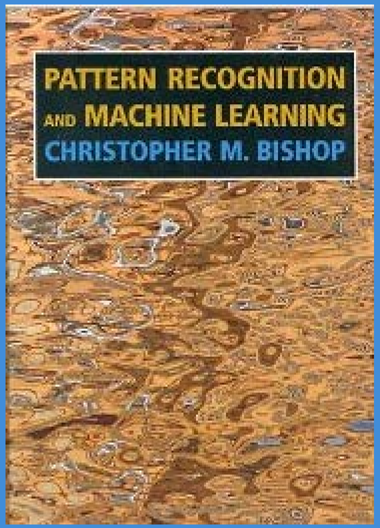
- Features are still conditionally independent
- Each feature has a Gaussian distribution given class

Further reading

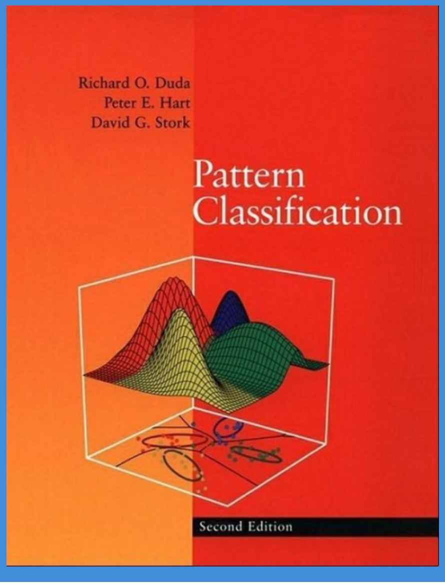
Manuscript (book chapters 1 and 2)
http://alex.smola.org/teaching/berkeley2012/slides/chapter1_2.pdf

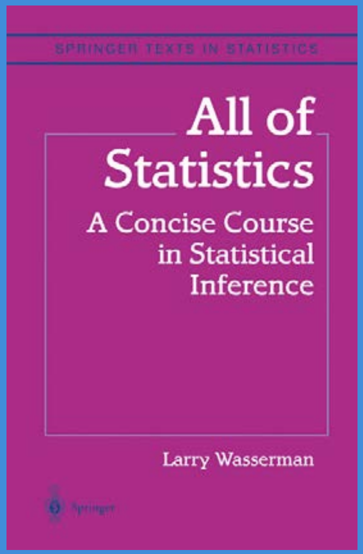
ML Books

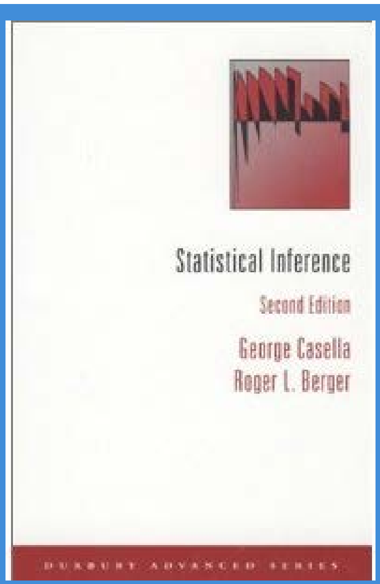




Statistics 101







A tiny bit of extra theory…

Feasible events = σ-algebra

Def: A collection of subsets of Ω is called a σ-algebra, denoted by $\mathcal{M}$, if it satisfies the following 3 properties:

a. $\emptyset \in \mathcal{M}$ (the empty set is an element of $\mathcal{M}$).

b. If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$
($\mathcal{M}$ is closed under complementation).

c. If $A_1, A_2, \ldots \in \mathcal{M}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{M}$
($\mathcal{M}$ is closed under countable unions).

Examples:

a. All subsets of Ω={1,2,3}: { $\emptyset$, {1},{2},{3},{1,2},{1,3},{2,3}, {1,2,3}}

b. $\Omega = (-\infty, \infty)$. $\mathcal{M} = \sigma((a,b)|a,b \in \mathbb{R})$ (Borel sets)

Measure

Let Ω be a set and $\mathcal{M}$ a σ-algebra over $\mathcal{M}$.

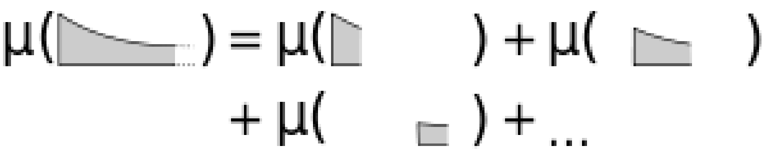
A function μ from $\mathcal{M}$ to $\mathbb{R} \cup \{\infty\}$ is called a measure if it satisfies the following properties.

(i) Nonnegativity. $\mu(A) \geq 0$ for each $A \in \mathcal{M}$.

(ii) $\exists E \in \mathcal{M}$ s.t. $\mu(E) = 0$, e.g. $\mu(\emptyset) = 0$.

(iii) σ-additivity: For disjoint sets $A_i \in \mathcal{M}$, we have

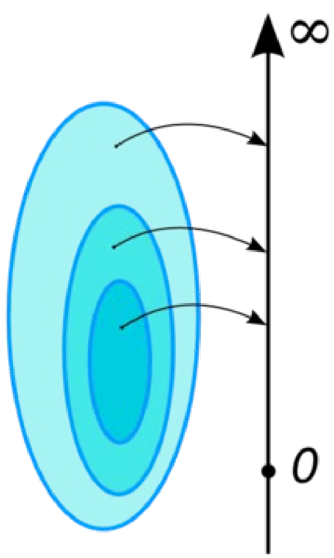
$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$



σ–additivity

Consequences:

Monotonity: $A_1 \subset A_2$, $A_1, A_2 \in \mathcal{M}$, then $\mu(A_1) \leq \mu(A_2)$.



monotonity

Important measures

Counting measure: $\mu(A) = |A|$, number of elements in the subset A.

Borel measure: $(\mathbb{R}, \mathcal{B} = \sigma((a,b)), \mu)$

$\mu((a,b)) = |b-a|$, length of the interval.

This is not a complete measure: There are Borel sets with zero measure, whose subsets are not Borel measurable…

Lebesgue measure: $(\mathbb{R}, \mathcal{L} \supset \mathcal{B}, \lambda)$

complete extension of the Borel measure, i.e. extension & every subset of every null set is Lebesgue measurable (having measure zero).

Lebesgue measure construction:

Given a subset A, its Lebesgue outer complete measure λ^* is defined as

$$\lambda^*(A) = \inf\{\mu(A) | A \subset B \in \mathcal{B}\}$$

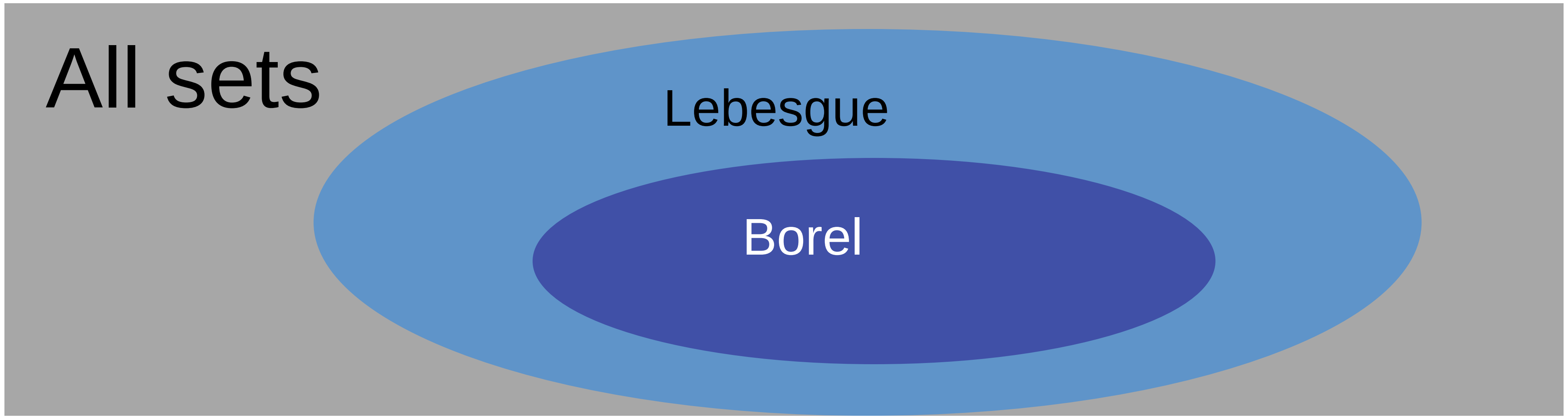
Def: $A \subset \mathbb{R}$ is Lebesgue measurable if for every $S \subset \mathbb{R}$

$$\lambda^*(S) = \lambda^*(S \cap A) + \lambda^*(S \setminus A)$$

Brain Teasers ☺

These might be surprising:

- Construct an uncountable Lebesgue set with measure zero.
- Construct a Lebesgue but not Borel set.
- Prove that there are not Lebesgue measurable sets. We can’t ask what is the probability of that event!
- Construct a Borel nullset who has a not measurable subset



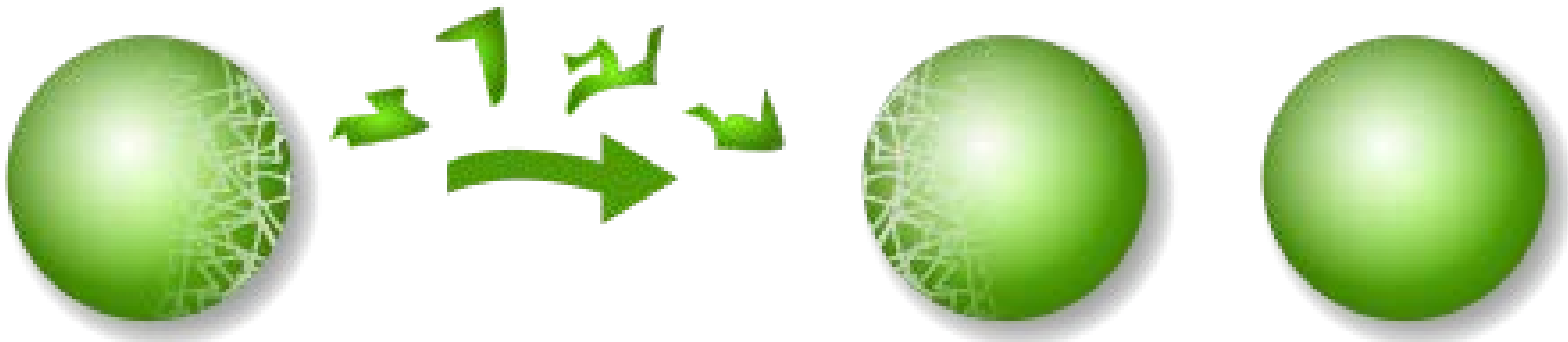
The Banach-Tarski paradox (1924)

Given a solid ball in 3-dimensional space, there exists a **decomposition** of the ball into a **finite** number of **non-overlapping** pieces (i.e., subsets), which can then be put back together in a different way to yield **two identical copies** of the original ball.

The reassembly process involves only **moving the pieces around and rotating them, without changing their shape**. However, the pieces themselves are not "solids" in the usual sense, but infinite scatterings of points.

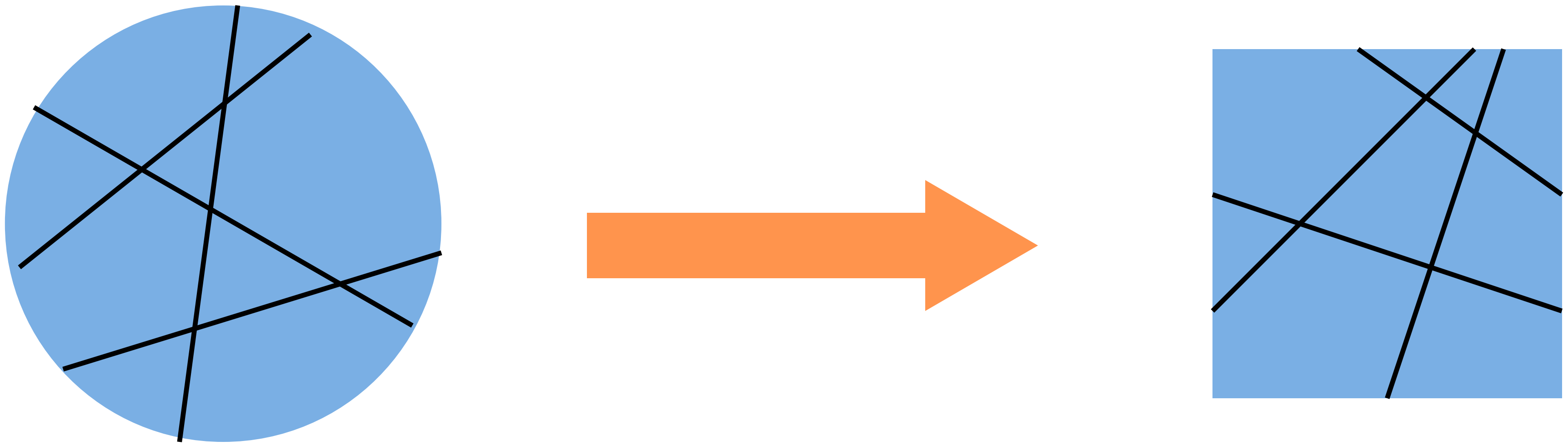
A stronger form of the theorem implies that given any two "reasonable" solid objects (such as a small ball and a huge ball), either one can be reassembled into the other.

This is often stated colloquially as "a pea can be chopped up and reassembled into the Sun."



Tarski's circle-squaring problem (1925)

Is it possible to take a disc in the plane, cut it into finitely many pieces, and reassemble the pieces so as to get a square of equal area?



Miklós Laczkovich (1990): It is possible using translations only; rotations are not required. It is not possible with scissors. The decomposition is non-constructive and uses about 10^{50} different pieces.

Thanks for your attention ☺

References

Many slides are recycled from

- Tom Mitchel
  http://www.cs.cmu.edu/~tom/10701_sp11/slides
- Alex Smola
- Aarti Singh
- Eric Xing
- Xi Chen
- http://www.math.ntu.edu.tw/~hchen/teaching/StatInference/notes/lecture2.pdf
- Wikipedia