SISE 9128: Introduction to Machine Learning

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Problem Sheet — Week 1

Teaching Period
October 8-19, 2001

The due date for these problems is Thursday, October 11

Problem 1 (SVD, Eigenvalues, and Positive Matrices)
Assume an arbitrary matrix $M \in \mathbb{R}^{m \times n}$ with $m \leq n$.

1. *Show that the matrix $MM^\top$ is positive semidefinite.*
2. *Show that the nonzero eigenvalues of $M^\top M$ and $MM^\top$ are identical. Hint: compute the eigevectors of $M^\top M$ from those of $MM^\top$.*
3. *Using the fact that there exist $U \in \mathbb{R}^{m \times m}, V \in \mathbb{R}^{m \times n}$, and a diagonal matrix $\Lambda \in \mathbb{R}^{m \times m}$ for which $M = U \Lambda V$, compute U, Λ, V using the eigenvalue/eigenvector decomposition of $M^\top M$ into $O^\top \Lambda O$. Here $O \in \mathbb{R}^{m \times m}$ is an orthogonal matrix and $\Lambda \in \mathbb{R}^{m \times m}$ is a diagonal matrix with only positive entries.*

Problem 2 (Vector Valued Functions)
Compute the first and second derivatives of the following functions

1. $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$ *where* $\mathbf{x}, \mathbf{c} \in \mathbb{R}^m$.
2. $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\top M \mathbf{x}$ *where* $\mathbf{x} \in \mathbb{R}^m$ *and* $M \in \mathbb{R}^m$. *What happens if* $M = M^\top$*?*
3. $f(X) = \operatorname{tr} MX$ *where* $M \in \mathbb{R}^{m \times n}$ *and* $X \in \mathbb{R}^{n \times m}$.
4. $f(\mathbf{x}) = g(\|\mathbf{x}_0 - \mathbf{x}\|)$ *where* $g : \mathbb{R}_0^+ \to \mathbb{R}$ *and* $\mathbf{x}_0, \mathbf{x} \in \mathbb{R}^m$.

Problem 3 (Dot Products of Smooth Functions)
Show that the following form is a dot product ($f, g : \mathbb{R} \to \mathbb{R}$)

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) g(x) dx + \int_{\mathbb{R}} f'(x) g'(x) dx.$$

Problem 4 (Hilbert Spaces and Derivatives)
Denote by $\mathcal{H}$ a Hilbert space and by $\langle \cdot, \cdot \rangle$ the dot products in $\mathcal{H}$.

For $f : \mathcal{H} \to \mathbb{R}$ with $f(x) = \frac{1}{2}\|x\|^2$ show that the Gateaux derivative $\frac{d}{dx} f$ is $\frac{d}{dx} f(x) = x$. Compute the second derivative (hint: this will be an operator).