ENGN 4520: Introduction to Machine Learning

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Problem Sheet — Week 1
The due date for these problems is Monday, May 7

Teaching Period
April 30 to June 8, 2001

A Theory (20 Points)

Problem 1 (SVD, Eigenvalues, and Positive Matrices, 6 Points)
Assume an arbitrary matrix $M \in \mathbb{R}^{m \times n}$ with $m \leq n$.

1. *Show that the matrix $MM^\top$ is positive semidefinite.*
2. *Show that the nonzero eigenvalues of $M^\top M$ and $MM^\top$ are identical. Hint: compute the eigevectors of $M^\top M$ from those of $MM^\top$.*
3. *Using the fact that there exist $U \in \mathbb{R}^{m \times m}, V \in \mathbb{R}^{m \times n}$, and a diagonal matrix $\Lambda \in \mathbb{R}^{m \times m}$ for which $M = U\Lambda V$, compute U, Λ, V using the eigenvalue/eigenvector decomposition of $M^\top M$ into $O^\top \Lambda O$. Here $O \in \mathbb{R}^{m \times m}$ is an orthogonal matrix and $\Lambda \in \mathbb{R}^{m \times m}$ is a diagonal matrix with only positive entries.*

Problem 2 (Vector Valued Functions, 8 Points)
Compute the first and second derivatives of the following functions

1. $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$ *where* $\mathbf{x}, \mathbf{c} \in \mathbb{R}^m$.
2. $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top M \mathbf{x}$ *where* $\mathbf{x} \in \mathbb{R}^m$ *and* $M \in \mathbb{R}^m$. *What happens if* $M = M^\top$*?*
3. $f(X) = \mathrm{tr}\ MX$ *where* $M \in \mathbb{R}^{m \times n}$ *and* $X \in \mathbb{R}^{n \times m}$.
4. $f(\mathbf{x}) = g(\|\mathbf{x}_0 - \mathbf{x}\|)$ *where* $g : \mathbb{R}_0^+ \to \mathbb{R}$ *and* $\mathbf{x}_0, \mathbf{x} \in \mathbb{R}^m$.

Problem 3 (Dot Products of Smooth Functions, 3 Points) *Show that the following form is a dot product ($f, g : \mathbb{R} \to \mathbb{R}$)*

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)g(x)dx + \int_{\mathbb{R}} f'(x)g'(x)dx.$$

Problem 4 (Hilbert Spaces and Derivatives, 3 Points)
Denote by $\mathcal{H}$ a Hilbert space and by $\langle \cdot, \cdot \rangle$ the dot products in $\mathcal{H}$.

For $f : \mathcal{H} \to \mathbb{R}$ with $f(x) = \frac{1}{2}\|x\|^2$ show that the Gateaux derivative $\frac{d}{dx}f$ is $\frac{d}{dx}f(x) = x$. Compute the second derivative (hint: this will be an operator).

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B Programming (20 Points)

Problem 5 (Cholesky Decomposition, 10 Points)

1. *Write a MATLAB or C/C++ program to decompose an arbitrary positive definite matrix* $M \in \mathbb{R}^{m \times m}$ *into* $M = R^\top R$ *where* $R \in \mathbb{R}^{m \times m}$ *is a lower triangular matrix. It should take as input* M *and output* R.
2. *Write a MATLAB or C/C++ program to solve the problem* $M\mathbf{x} = \mathbf{y}$ *for a given* $\mathbf{y} \in \mathbb{R}^m$. *Hint, solve* $R^\top \mathbf{x}' = \mathbf{y}$ *and then* $R\mathbf{x} = \mathbf{x}'$. *It should take as input* R *and* $\mathbf{x}$ *and output* $\mathbf{y}$.
3. *What happens if* M *does not have full rank (***bonus question***)?*

Hint: *you can check your results using the* `chol` *routine of MATLAB.*

Problem 6 (Function Minimization by Interval Cutting, 10 Points)

1. *Write a MATLAB or C/C++ program to minimize convex functions via interval cutting. It should take as inputs* f, f', *the initial interval* $[a, b]$, *where the minimum can be found, and a precision* ϵ *and return* $x_{\min}, f(x_{\min})$, *and* $f'(x_{\min})$.
2. *Minimize the function* $f(x) = e^{-x} + x^4 + 3(x-10)^2$. *Hint: the minimum lies in* $[-10, 10]$.
3. *Plot the function values* $f\left(\frac{A+B}{2}\right)$ *for* f *defined as above.*

Hint: *you can check your results using the* `fminbnd` *routine of MATLAB.*