1 Probability Inequalities [Jing; 25 pts]

(a)

\[ P(\bar{X}_n \geq 0.5) \leq \frac{E[\bar{X}_n]}{0.5} \leq \frac{0.2}{0.5} \leq \frac{2}{5} \]

(b) First, calculate the variance of a bernoulli random variable: \( Var(X) = p(1-p) \).

\[ Var(X_i) = 0.2(1-0.2) \]
\[ Var(X_i) = 0.2(0.8) \]
\[ Var(X_i) = 0.16 = \frac{4}{25} \]

Now, find the bound.

\[ P(|\bar{X}_n - E[\bar{X}_n]| \geq k) \leq \frac{Var[\bar{X}_n]}{k^2} \]
\[ P(|\bar{X}_n - 0.2| \geq 0.3) \leq \frac{Var[\frac{1}{n} \sum_{i=1}^{n} X_i]}{(0.3)^2} \]
\[ P(\bar{X}_n \geq 0.5) \leq \frac{(\frac{1}{n})^2 \sum_{i=1}^{n} Var(X_i)}{0.09} \]
\[ P(\bar{X}_n \geq 0.5) \leq \frac{(\frac{1}{n})^2 n(0.16)}{0.09} \]
\[ P(\bar{X}_n \geq 0.5) \leq \frac{100 \cdot 4}{9n \cdot 25} \]
\[ P(\bar{X}_n \geq 0.5) \leq \frac{16}{9n} \]

(c)

\[ P(|\bar{X}_n - E[\bar{X}_n]| \geq \epsilon) \leq 2e^{-2n\epsilon^2/(b-a)^2}c \]
\[ P(|\bar{X}_n - 0.2| \geq 0.3) \leq 2e^{-2n(0.3)^2/(1-0)^2} \]
\[ P(\bar{X}_n \geq 0.5) \leq 2e^{-\frac{2n}{100}} \]
\[ P(\bar{X}_n \geq 0.5) \leq 2e^{-\frac{2n}{200}} \]
(d) and (e)

![Graph showing the comparison of original space, Markov, Chebychev, Hoeffding, and Exact methods for the number of coin flips and probability.](image-url)
(f)
(g)

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<th>log(Number of coin flips)</th>
<th>log(Probability)</th>
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Markov
Chebychev
Hoeffding
Exact

(h)

**Original Space**

- Markov is linear, has a \( y = b \) relationship, \( y = \frac{2}{5} \).

**Semi-log Space**

- Markov is linear, has \( y = \log(b) \), which is still a straight line.
- Hoeffding is linear, because taking the log of an exponential will get you a linear \( y = ax \) relationship, where \( a = -9/50 \).

**log-log Space**

- Markov is still linear.
- Chebychev is linear, because \( y = x^b \) is a line with slope \( b = -1 \)