Bayes rule

- recall def of conditional:
  - P(a|b) = P(a^b) / P(b) if P(b) != 0

$P(a|b)P(b) = P(a \wedge b) = P(b|a)P(a)$

$P(a|b) = P(b|a)P(a) / P(b)$

$P(b) \neq 0$

mult thru by P(b): P(a|b) P(b) = P(a^b)
holds even if P(b)=0 (check), so some take this as basic def of conditioning

so: P(a^b) = P(a|b)P(b) = P(b|a)P(a)
Bayes rule: divide last eq by P(b) (if P(b) != 0)
P(a|b) = P(b|a)P(a)/P(b)
note: if b is observed, don't need to worry about P(b) == 0
good rule: only condition on events we might observe :-)
Bayes rule w/ background event C (just like changing to a smaller universe)
P(a^b|C) = P(a|b,C)P(b|C) = P(b|a,C)P(a|C)

Bayes rule: sum version

- P(a | b) = P(b | a) P(a) / P(b)

what if we don't know P(b), but still know P(b|a) and P(a)
(often happens)
suppose MEEP A = a1, a2, ..., an for moderate n

we know sum_i P(ai | b) = 1
so sum_i P(ai | b) P(b) = P(b) (mult by P(b) on both sides)
LHS is sum_i P(b|ai) P(ai) (another application of Bayes rule)
each term assumed known, sum tractable since n moderate

Bayes rule in ML

- P(model | data) = P(data | model) P(model) / P(data)

Why do we care about Bayes? Lets us take info about conditional in one direction (P(data | model)) and get info about the other direction (P(model| data)).

LHS P(model|data) tells us which models are probable give the data. This is (mostly) what we want out of ML! Called “posterior”

P(data | model) usually has an explicit formula, e.g., gaussian distribution: P(x_i|mu) = exp(-(x_i-mu)^2/2)/√2pi). Called “likelihood”

P(model) = “prior” -- sometimes controversial but not hard to write a minimally reasonable one

P(data): “normalizing constant” (also “annoyance”) -- often hard to get
- can use sum version of Bayes rule (sum numerator over models)
  - OK if not too many models under consideration
- can use sum version + approximation tricks (numerical quadrature, MCMC like Gibbs)
- another idea on next slide

note: normalizing constants are often ignored. This is a common pattern, but it doesn’t mean it’s always safe (or a good idea) to ignore them! Sometimes all of the useful information is in the normalizing constant...

Bayes rule vs. MAP vs. MLE

- P(model | data) = P(data | model) P(model) / P(data)

MAP: just take model for which RHS is highest -- if we have to choose just one point from posterior density, why not this one?

[sketch]

now P(data) really is ignorable (doesn't change order)

Seems like a horrible approximation (from Bayesian perspective), but actually has theory to back it up.

MLE: ignore P(model) term too. Why? Because people argue about the prior; because with enough evidence from data it might become negligible.

Seems even worse, but still has theory to back it up. (see next slide)

Frequentist vs. Bayes

Jerzy Neyman

rev. Thomas Bayes

see also: http://www.xkcd.com/1132/

- Nature as adversary vs. Nature as probability distribution
- Probability as long-run frequency of repeatable events vs. odds for bets I'm willing to take

treat Nature as a probability distribution vs. an adversary
Bayes: if you have the right distribution over models and can compute with it, good things happen
Frequentist: but both of those are horrible assumptions
Plenty of paradoxes for both sides if you want to cast stones

Test for a rare disease

- About 0.1% of all people are infected
- Test detects **all** infections
- Test is highly specific: 1% false positive
- You test positive. What is the probability you have the disease?

P(disease)
P(test | +disease)
P(test | −disease)

P(+disease | +test)
= P(+test | +disease) P(+disease) / P(+test)
P(+t) = P(+t|+d)P(+d) + P(+t|−d)P(−d)
= 1 * .001 / (1*.001 + .01*.999)
= .091

Test for a rare disease

- About 0.1% of all people are infected

Bonus: what is probability an average med student gets this question wrong?

the disease?

P(disease)
P(test | +disease)
P(test | −disease)

P(+disease | +test)
= P(+test | +disease) P(+disease) / P(+test)
P(+t) = P(+t|+d)P(+d) + P(+t|−d)P(−d)
= 1 * .001 / (1*.001 + .01*.999)
= .091

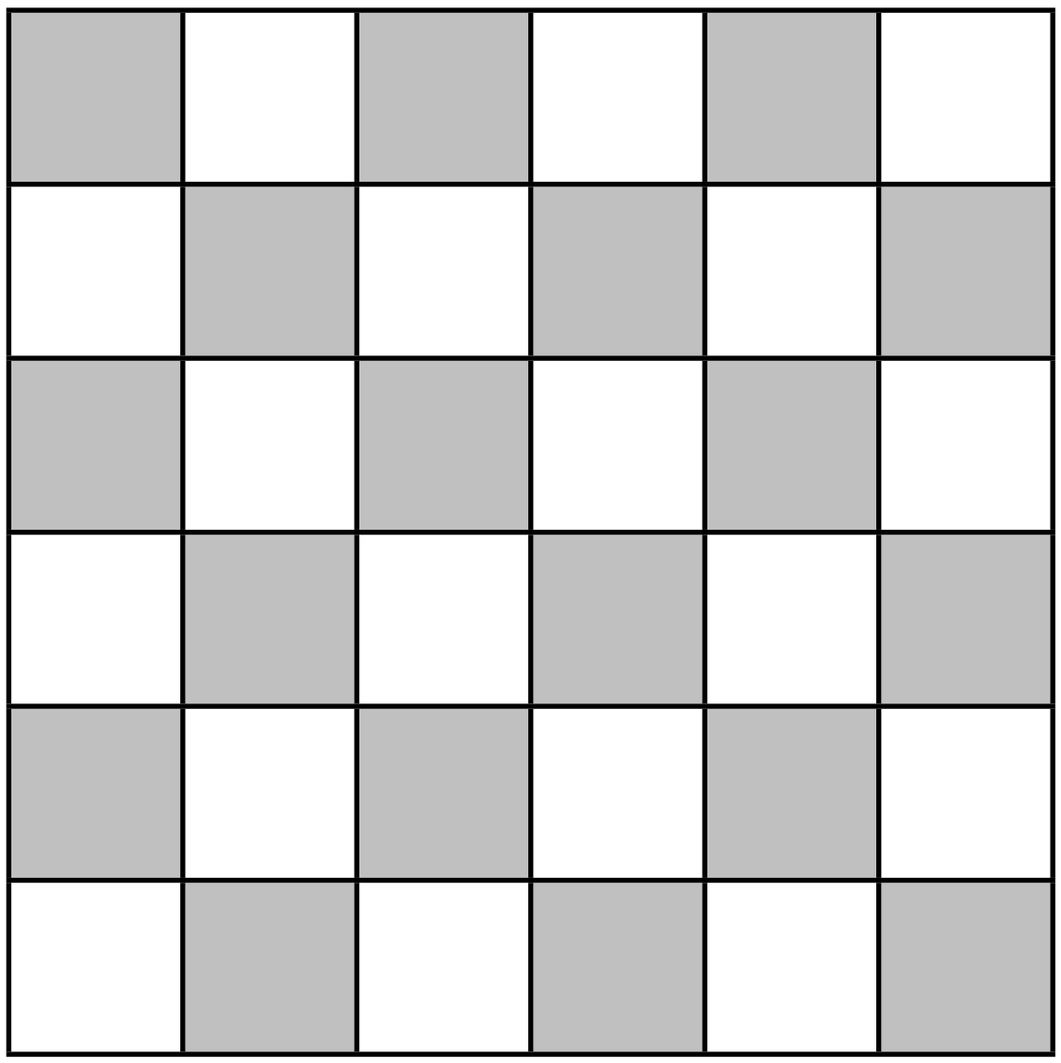
Follow-up test

- Test 2: detects 90% of infections, 5% false positives
  - P(+disease | +test1, +test2) =

P(+disease | +test1, +test2) =
P(+1, +2 | +d) P(+d) / P(+1, +2)
P(+12) = P(+12|+d)P(+d) + P(+12|-d)P(-d)
= 1*.9*.001 / (1*.9*.001 + .05*.01*.999)
= .643

Test 1 seems better than test 2 -- why not use test 1 twice?
A: T1 is conditionally independent of T2 given disease state -- probably not true for T1 w/ itself

Independence



for events: defined as P(a^b) = P(a)P(b), like rows/cols of checkerboard
P(~a^b) = P(b) – P(a^b) = P(b) – P(a)P(b) = (1–P(a))P(b) = P(~a)P(b)

for r.v.s: P(A,B) = P(A)P(B)
    shorthand: joint probability table = outer product of marginal probability tables
    P(A=a_i ^ B=b_i) = P(A=a_i) P(B=b_i)

intuition: knowing a or ~a tells us nothing about B
Bayes rule version: P(A) = P(A|B)
    i.e., P(A=a_i) = P(A=a_i | B=b_j) for all a_i, b_j w/ P(B=b_j > 0)
    follows: P(a|b) = P(a^b)/P(b) = P(a)P(b)/P(b) = P(a), as long as P(b)!=0
    some take this as definition of independence

Conditional independence

conditional independence: X _|_ Y | Z if, for all values of z_i, P(X,Y|z_i)=P(X|z_i)P(Y|z_i)
i.e., after we condition on z_i, X and Y are independent

P(x, y | z) = P(x | z) P(y | z) for any values xyz of XYZ
this is a statement about a 3d table and two 2d tables

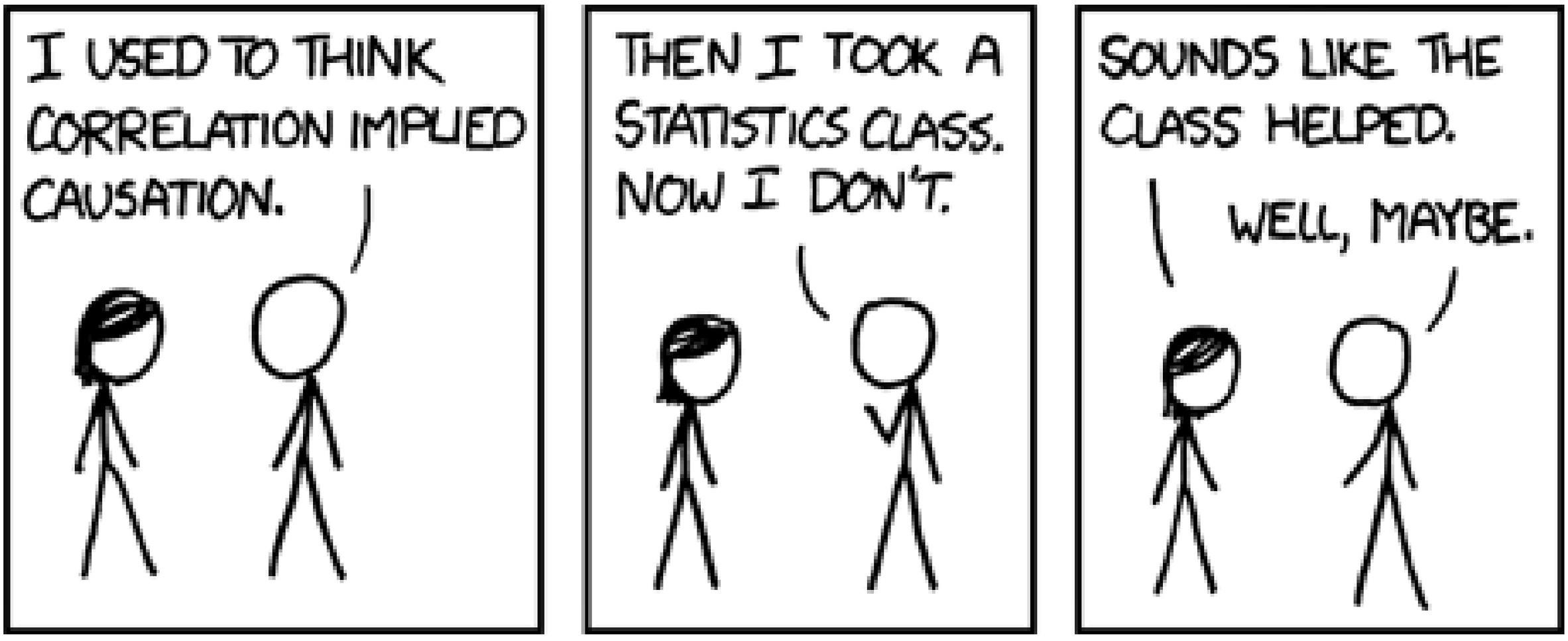
Conditionally Independent

London taxi drivers: A survey has pointed out a positive and significant correlation between the number of accidents and wearing coats. They concluded that coats could hinder movements of drivers and be the cause of accidents. A new law was prepared to prohibit drivers from wearing coats when driving.

Finally another study pointed out that people wear coats when it rains…

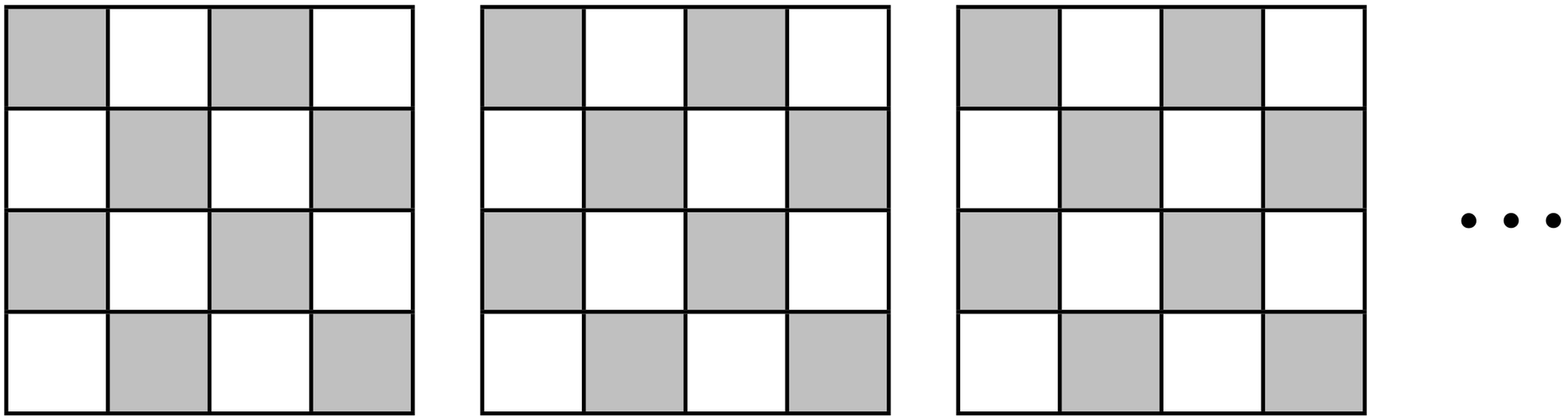
slide credit: Barnabas

More on the importance of conditioning



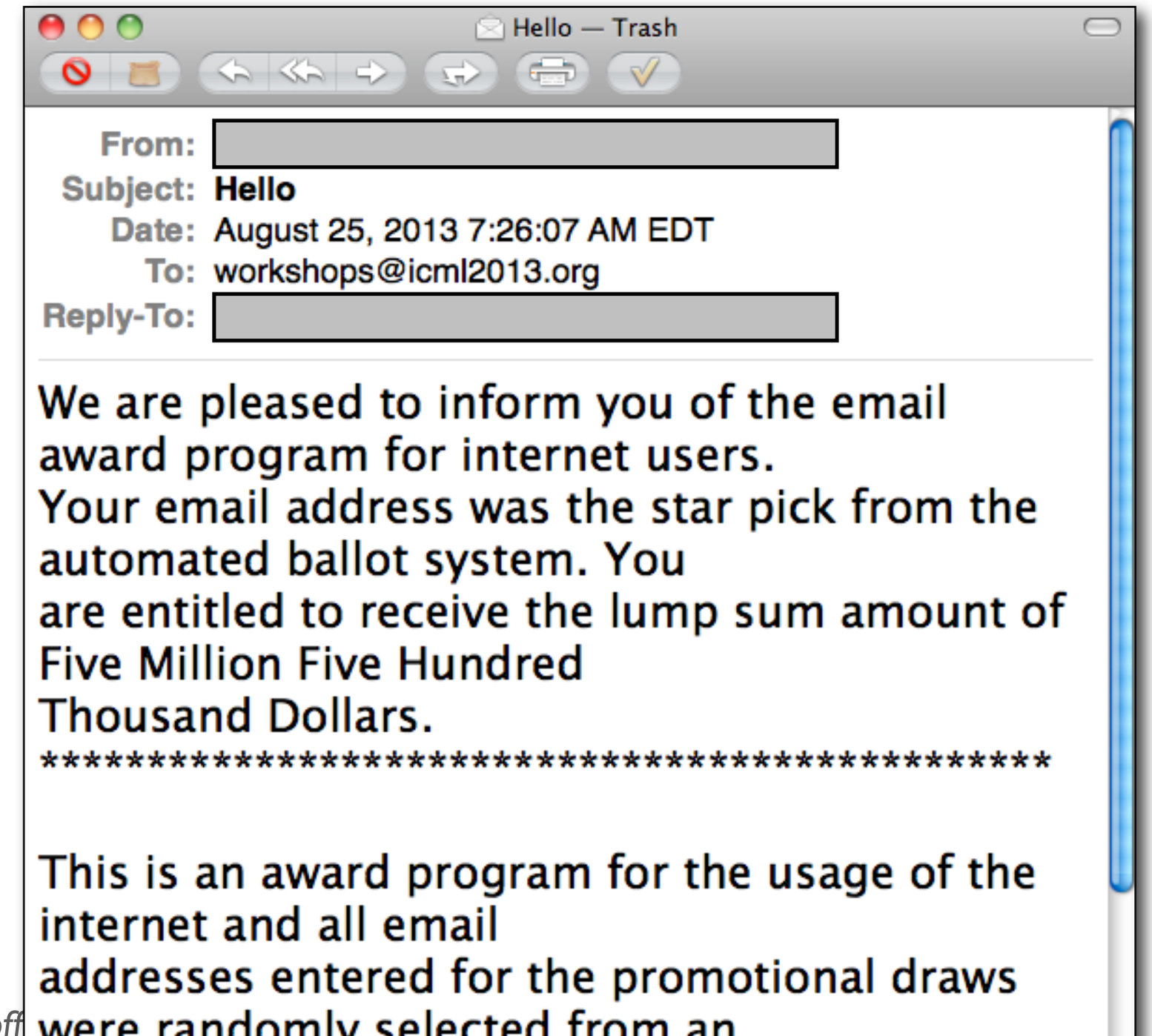
humor credit: xkcd

Samples



i.i.d. sample of an r.v.: N independent copies of the same checkerboard
sample is itself an r.v. in a bigger space (a 2N-dim hypercheckerboard)
    atomic events are tuples of original atomic events
sample (r.v.) vs. population (the One True checkerboard)
statistic = any function of the sample
    usu. order-independent (symmetric) but doesn't have to be
    statistics are r.v.s
    e.g., sample mean, variance
goal of statistics (big or small S): use statistics to find out something about the population

Recall: spam filtering



classification problem: given data (x_i, y_i)
N pairs, x_i \in {0,1}^d, y_i \in {0,1} -- write x_i = (x_{i1} .. x_{id})

produce rule which goes from future x -> predicted y
e.g.: spam filtering: x_ij = presence of word i in doc j

bag of words

Bag of words

A ridiculously naive assumption

- Assume:

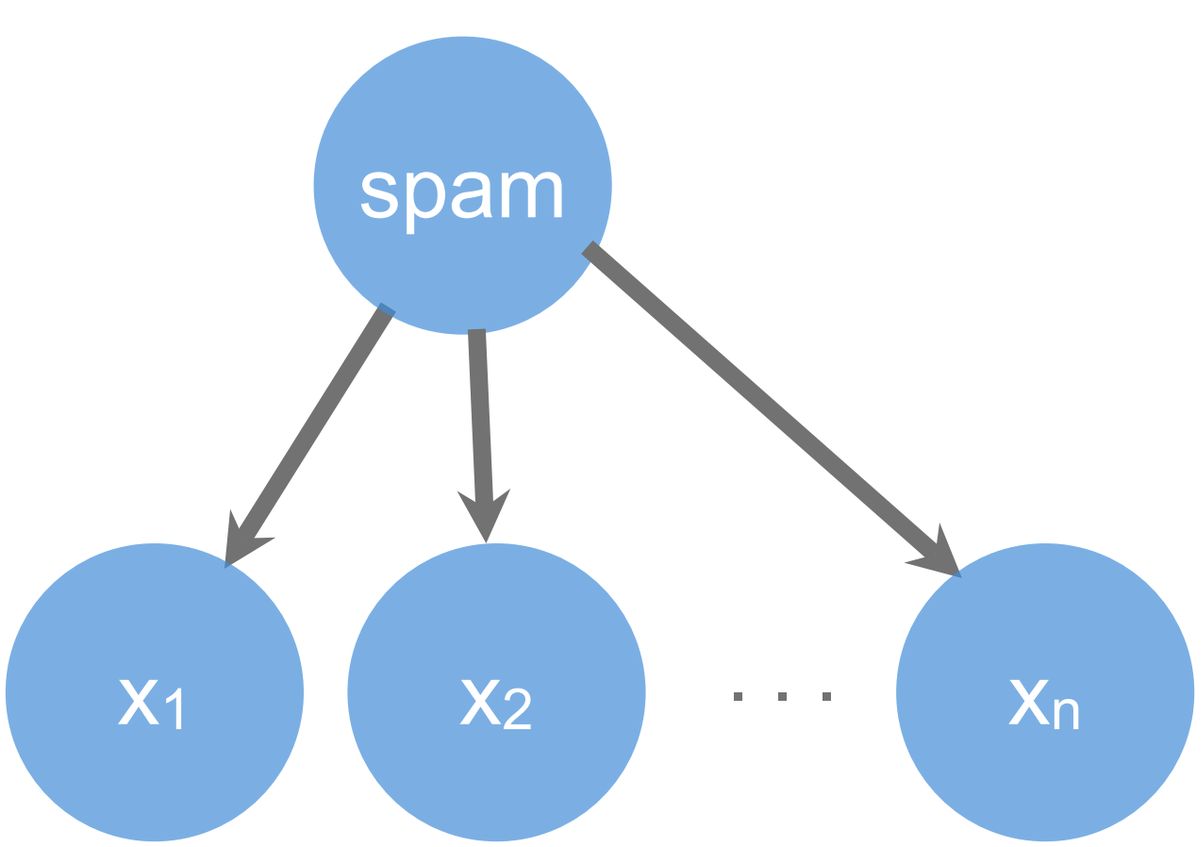
- Clearly false:
- Given this assumption, use Bayes rule

assumption: x_{ij} _|_ x_{ik} | c_i
for all j,k \in 1..d, j != k

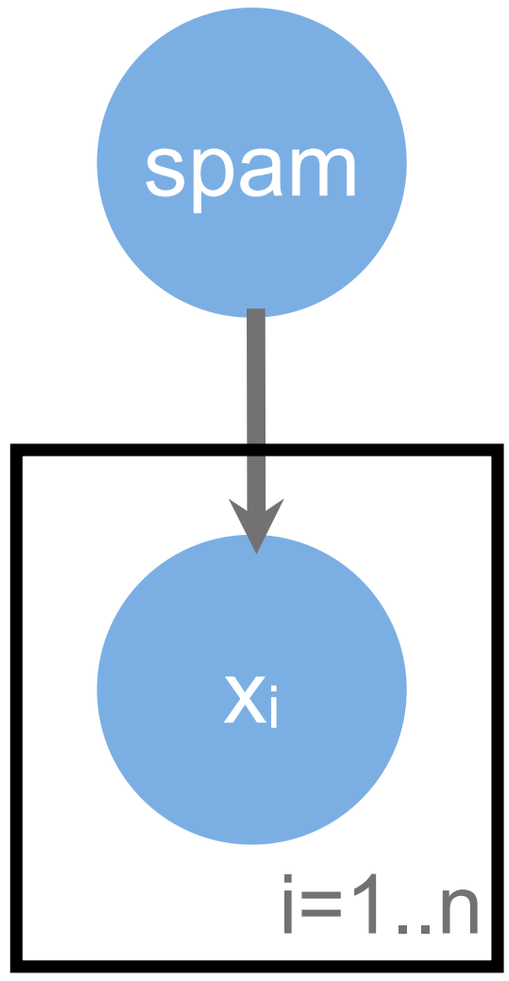
clearly false: "CMU" not independent of "Bayes"

given this "naive" assumption, use "Bayes" rule

Graphical model



$$P(X_1, \ldots, X_d \mid \text{spam}) = \prod_{i=1}^{d} P(X_i|\text{spam})$$



arrows spam-> xi say xi depends on spam
lack of other arrows into xi: the xi's are conditionally independent given spam

"macro" or "for loop" shorthand: called a "plate model"

Naive Bayes

- P(spam | email ∧ award ∧ program ∧ for ∧ internet ∧ users ∧ lump ∧ sum ∧ of ∧ Five ∧ Million)

P(spam | email award program for internet users lump sum of Five Million)
= P(email ... Million | spam) P(spam) /
  [P(email ... Million | spam) P(spam) + P(email ... Million | not-spam) P(~spam)]
                (sum version of Bayes rule)
= P(email | spam) P(award | spam) ... P(Million | spam) P(spam) /
  ["" | spam + "" | ~spam]
                (independence assumption)

suppose we know P(word j | spam) and P(word j | ~spam) for all j
and suppose we know P(spam) and P(~spam)
        how? see slightly later

then above is easy to calculate!
now keep messages w/ P(spam) < threshold

adjust threshold based on user preference: chance of missing internet lottery win vs. wants to get work done

In log space

z_{spam} = ln(P(email | spam) P(award | spam) ... P(Million | spam) P(spam))

$z_{\sim spam}$ = ln(P(email | ~spam) ... P(Million | ~spam) P(~spam))

z_spam = ln(P(email | spam) P(award | spam) ... P(Million | spam) P(spam))
z_~spam = ln(P(email | ~spam) P(award | ~spam) ... P(Million | ~spam) P(~spam))

result is then
    P(spam | ...) = exp(z_spam) / [exp(z_spam) + exp(z_~spam)]
    = 1 / [1 + exp(z_~spam – z_spam)]
    = 1 / [1 + exp(–z)]
where z = z_spam – z_~spam

sigmoid or logistic function (sketch): logit(z)
big z: confident in spam
big –z: confident in ~spam

Collect terms

z_{spam} = ln(P(email | spam) P(award | spam) ... P(Million | spam) P(spam))

$z_{\sim spam}$ = ln(P(email | ~spam) ... P(Million | ~spam) P(~spam))

$z = z_{spam} - z_{spam}$

= ln P(spam) - ln(P(~spam)) <-- call this b
+ ln P(email | spam) - ln P(email | ~spam) <-- call this w_{email}
+ ln P(award | spam) - ln P(award | ~spam) <-- w_{award}
+ ...

sum is over words in our message
i.e.: to classify message x_i, compute b + sum_{j\in vocabulary} w_j x_{ij}
    (x_{ij} = 0 when word j not in message)
threshold on result: b + sum_j w_j x_{ij} >= 0
    "linear discriminant"
"decision boundary" : b + sum_j w_j x_{ij} = 0

Linear discriminant

threshold on result: b + sum_j w_j x_{ij} >= 0
"linear discriminant"
"decision boundary" : b + sum_j w_j x_{ij} = 0

Intuitions

word of warning: nominally, P(spam) is logit(z)
really, logit(z) usually close to 0 or 1 even in marginal cases
why? [failure of independence assumption]
fix: use logit(eps*z) for appropriately-chosen eps (***)

what are highly useful/discriminative words?
    log(P(word j|spam)/P(word j|~spam)) far from zero
    i.e., really low probability one class, moderate to high in the other
        (since log is much more sensitive near 0)
    but decent chance of actually seeing word j
suggested measure: P(j)log(P(j|s)/P(j|~s)) -- looks like KL / relative entropy!
eg: "lottery" really low in legit emails -- not high in spam, but significantly nonzero

How to get probabilities?

How to get the needed probabilities?
- flip coin, count heads or tails
- 3H, 7T: estimate P(H)=.3
- what if 0H, 2T?
- hard zero probability seems bad
- Hack: estimate P(H) = (#H+0.5)/(#H+#T+1)
  - here, 0.5/3 = 1/6: pretty sure but not hard zero
- turns out not to be a hack after all…
- called "Laplace smoothing", MAP for Dirichlet prior

for words, sample count: might expect it to be proportional to population prob
- estimator: count(word)/(total # words)

e.g., if we see "avocado" 10 times in 100k words, estimate P(avocado) = 1/10k

count 0: e.g., "syzygy" not in training set but appears in test doc
- get log(0)=-infty in answer, bad.

estimate p(word) as (lambda+count(word))/(lambda*words + total#words)
- small lambda > 0

Improvements

n-grams, character string features, treat (multiple) headers vs. body differently, features of attachments, take account of body length, generalize among words using edit distance, stemming, POS-tagging, parsing, named entity resolution, collaborative spam-filters (if everyone's winning the lottery...), soft whitelists/blacklists (of domains, IP addresses, senders), ...

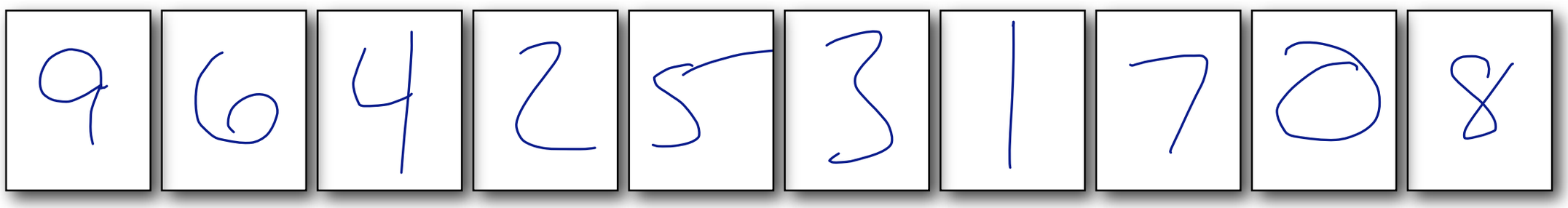
general rule: most important part of practical ML is to think of good sources of information, then figure out how to get algorithm to pay attention to them

Perceptron

classification again:
    X = ((x1, y1) .. (xN, yN))
    yi \in {-1, 1} <-- instead of 0/1 for convenience
    assume x_ij in {0,1} if desired, but works just as well w/ \Re^d

ex: spam filtering, or digit classific'n

Digit recognition



digit recognition:

x_{ij} = pixel j intensity in image i

classify (say) 4 vs 9

Linear separability

linear discriminant: z_i = x_i \cdot w + b
(note notation change: x_i\cdot w = \sum_j x_ij w_j; Alex wrote <xi,w>)

z_i > 0 when y_i = +1
z_i < 0 when y_i = −1
i.e., no errors and nothing on decision boundary

suppose data set X is separable; how can we find w in this case?

Simplify notation

simplify notation: homogeneous rep:
    xi \in \Re^d
    assume last component x_{id} = 1 in all examples
    if not, set d := d+1 and just append 1 to every example
    now, add b to last component of w: say w = w_old + [0 0 0 ... b]
    x_i \cdot w = x_i\cdot (w_old + [0 0 0 ... b])
     = x_i \cdot w_old + b = z_i
    benefit: don't need b in our notation any more

flip -ves:
    write ui = yi xi \in \Re^n
    x_i \cdot w > 0 when y_i = +1
        ==> u_i \cdot w > 0
    x_i \cdot w < 0 when y_i = -1
        ==> u_i \cdot w > 0
    so now, just write u_i \cdot w > 0 for all i

First algorithm: LP

LP: we have a system of linear inequalities
almost an LP -- if it were, we'd be done (hand it to CPLEX and go for coffee)
but our ineqs are strict (LP software requires >= not >)
suppose we know eps = min_i u_i\cdot w
now we can write u_i\cdot w >= eps for all i
but we don't know eps -- trick: let v = w/eps
now u_i\cdot v >= 1 for all i
solve for v -- and eps doesn't matter

remember this algorithm: we'll see another one like it when we get to support vector machines

#2: the "perceptron algorithm"

$$u_i \cdot w > 0 \qquad \forall i$$

simple algorithm:
    start at w = 0
    pick a random training point ui = yi xi
    if inequality is satisfied, great
    if not, w := w + ui
        ui . w_new = ui . w + ui . ui > ui . w
        closer to satisfying ineq
    repeat

This algorithm was implemented in 1958 on the "Mark I Perceptron", which represented weights with variable resistors (i.e., dimmer switches), and used motors to turn the knobs for the update (!)

To analyze: draw a sphere around all ui. (For convenience suppose radius 1; if not, renorm.)

Margin gamma: max_{||w|| <= 1} min_i u_i . w (draw it)

Perceptron proof

suppose w* (unknown) separates w/ margin gamma, ||w*||=1

On every update,
    w_new . w* = (w + ui) . w*
    = w . w* + ui . w*
    >= w . w* + gamma
So, after T mistakes, w . w* >= T gamma
    (starts at 0, increases by gamma each time)

w_new . w_new = (w + ui) . (w + ui)
    = w . w + 2 w . ui + ui . ui
    <= w . w + 1
So, after T mistakes, w . w <= T

T gamma <= w . w*
    <= ||w|| ||w*|| (Cauchy-Schwartz)
    = sqrt(w.w)
    <= sqrt(T)
divide thru by sqrt T and by gamma:
sqrt(T) <= 1/gamma