Generalization Performance of Regularization Networks and Support Vector Machines Via Entropy Numbers of Compact Operators

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***Abstract*—We derive new bounds for the generalization error of kernel machines, such as support vector machines and related regularization networks by obtaining new bounds on their covering numbers. The proofs make use of a viewpoint that is apparently novel in the field of statistical learning theory. The hypothesis class is described in terms of a linear operator mapping from a possibly infinite-dimensional unit ball in feature space into a finite-dimensional space. The covering numbers of the class are then determined via the entropy numbers of the operator. These numbers, which characterize the degree of compactness of the operator, can be bounded in terms of the eigenvalues of an integral operator induced by the kernel function used by the machine. As a consequence, we are able to theoretically explain the effect of the choice of kernel function on the generalization performance of support vector machines.**

***Index Terms*—Covering numbers, ϵ-entropy, kernel methods, linear operators, metric entropy, statistical learning theory, support vector (SV) machines.**

I. Introduction

In this paper we give new bounds on the covering numbers for kernel machines. This leads to improved bounds on their generalization performance. Kernel machines perform a mapping from input space into a feature space (see, e.g., [1], [34]), construct regression functions or decision boundaries based on this mapping, and use constraints in feature space for capacity control. Support vector (SV) machines, which have recently been proposed as a new class of learning algorithms solving problems of pattern recognition, regression estimation, and operator inversion [53] are a well-known example of this class. We will use SV machines as our model of choice to show how bounds on the covering numbers can be obtained. We outline the relatively standard methods one can then use to hence bound their generalization performance. SV machines, like most kernel-based methods, possess the nice property of defining the feature map in a manner that allows its computation implicitly at little additional computational cost. Our reasoning also applies to similar algorithms such as regularization networks [16] or certain unsupervised learning algorithms [41]. Let us now take a closer look at SV machines. Central to them are two ideas: capacity control by maximizing margins, and the use of nonlinear kernel functions.

A. Capacity Control

In order to perform pattern recognition using linear hyperplanes, often a maximum margin of separation between the classes is sought, as this leads to good generalization ability independent of the dimensionality [55], [53], [43]. It can be shown that for separable training data

$$(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_m, y_m) \in \mathbb{R}^d \times \{\pm 1\} \tag{1}$$

this is achieved by minimizing $||\boldsymbol{w}||_2$ subject to the constraints $y_j(\langle \boldsymbol{w}, \boldsymbol{x}_j \rangle + b) \geq 1$ for $j = 1, \ldots, m$, and some $b \in \mathbb{R}$. The decision function then takes the form

$$f(\boldsymbol{x}) = \operatorname{sgn}(\langle \boldsymbol{w}, \boldsymbol{x} \rangle + b). \tag{2}$$

Similarly, a linear regression

$$f(\boldsymbol{x}) = \langle \boldsymbol{w}, \boldsymbol{x} \rangle + b \tag{3}$$

can be estimated from data

$$(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_m, y_m) \in \mathbb{R}^d \times \mathbb{R} \tag{4}$$

by finding the flattest function which approximates the data within some margin of error: in this case, one minimizes $||\boldsymbol{w}||_2$ subject to $|f(\boldsymbol{x}_j) - y_j| \leq \varepsilon$, where the parameter $\varepsilon > 0$ plays the role of the margin, albeit not in the space of the inputs $\boldsymbol{x}$, but in that of the outputs y.

In both cases, generalizations for the nonseparable or nonrealizable case exist, using various types of cost functions [14], [53], [46].

B. Nonlinear Kernels

In order to apply the above reasoning to a rather general class of *nonlinear* functions, one can use kernels computing dot products in high-dimensional spaces nonlinearly related to input space [1], [10]. Under certain conditions on a kernel k, to be stated below (Theorem 4), there exists a nonlinear map Φ into a

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reproducing kernel Hilbert space F (see, e.g., [40]) such that k computes the dot product in F, i.e.,

$$k(\boldsymbol{x}, \boldsymbol{y}) = \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{y}) \rangle_F. \tag{5}$$

Given any algorithm which can be expressed in terms of dot products exclusively, one can thus construct a nonlinear version of it by substituting a kernel for the dot product. Examples of such machines include SV pattern recognition [10], SV regression estimation [53], and kernel principal component analysis [41].

By using the kernel trick for SV machines, the maximum margin idea is thus extended to a large variety of nonlinear function classes (e.g., radial basis function networks, polynomial networks, neural networks), which in the case of regression estimation comprise functions written as kernel expansions

$$f(\boldsymbol{x}) = \sum_{i=1}^{m} \alpha_j k(\boldsymbol{x}_j, \boldsymbol{x}) + b \tag{6}$$

with $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, m$. It has been noticed that different kernels can be characterized by their regularization properties [48]: SV machines are regularization networks minimizing the regularized risk

$$R_{\mathrm{reg}}[f] = R_{\mathrm{emp}}[f] + \frac{\lambda}{2} \|Pf\|^2$$

(with a regularization parameter $\lambda \geq 0$, and a regularization operator P) over the set of functions of the form (6), provided that k and P are interrelated by

$$k(\boldsymbol{x}_s, \boldsymbol{x}_t) = \langle (Pk)(\boldsymbol{x}_s, \cdot), (Pk)(\boldsymbol{x}_t, \cdot) \rangle.$$

To this end, k is chosen as a Green's function of P^*P where P^* is the adjoint of P.

This provides insight into the regularization properties of SV kernels. However, it does not completely settle the issue of how to select a kernel for a given learning problem, and how using a specific kernel might influence the performance of an SV machine.

C. Outline of the Paper

In the present work, we show that properties of the spectrum of the kernel can be used to make statements about the generalization error of the associated class of learning machines. Unlike in previous SV learning studies, the kernel is no longer merely a means of broadening the class of functions used, e.g., by making a nonseparable dataset separable in a feature space nonlinearly related to input space. Rather, we now view it as a constructive handle by which we can control the generalization error.

A key feature of the present paper is the manner in which we *directly* bound the covering numbers of interest rather than making use of a combinatorial dimension (such as the Vapnik–Chervonenkis (VC) dimension or the fat-shattering dimension) and subsequent application of a general result relating such dimensions to covering numbers. We bound covering numbers directly by viewing the values induced by the relevant class of functions as the image of a unit ball under a particular compact operator. A general overview of the method is given in Section III.

The remainder of the paper is organized as follows. We start by introducing notation and definitions (Section II). Section IV formulates generalization error bounds in terms of covering numbers. Section V contains the main result bounding entropy numbers in terms of the spectrum of a given kernel. The results in this paper rest on a connection between covering numbers of function classes and entropy numbers of suitably defined operators. In particular, we derive an upper bound on the entropy numbers in terms of the size of the weight vector in feature space and the eigenvalues of the kernel used. Section VI shows how to make use of kernels such as $k(x, y) = e^{-(x-y)^2}$ which do not have a discrete spectrum. Section VII presents some results on the entropy numbers obtained for given rates of decay of eigenvalues and Section VIII shows how to extend the results to several dimensions. The concluding section (Section IX) indicates how the various results in the paper can be glued together in order to obtain overall bounds on the generalization error. All of the examples we provide for the calculation of eigenvalues are for translation-invariant kernels (i.e., convolutional kernels); this is merely for convenience—the general theory is not restricted to such kernels. Key new results are labeled as propositions.

We do not present a single master generalization error theorem for four key reasons: 1) the only novelty in the paper lies in the computation of covering numbers themselves; 2) the particular statistical result one needs to use depends on the specific problem situation; 3) many of the results obtained are in a form which, while quite amenable to ready computation on a computer, do not provide much direct insight by merely looking at them, except perhaps in the asymptotic sense; and, finally, 4) some applications (such as classification) where further quantities like margins are estimated in a data dependent fashion, need an additional luckiness argument [44] to apply the bounds.

Thus, although our goal has been theorems, we are ultimately forced to resort to a computer to make use of our results. This is not necessarily a disadvantage—it is both a strength and a weakness of structural risk minimization (SRM) [56] that a good generalization error bound is both necessary and sufficient to make the method work well. In [20], some more explicit formulas based on the present work and more suitable for SRM are developed.

II. Definitions and Notation

For $d \in \mathbb{N}$, $\mathbb{R}^d$ denotes the d-dimensional space of vectors $\boldsymbol{x} = (x_1, \ldots, x_d)^T$. We define spaces ℓ_p^d as follows: as vector spaces, they are identical to $\mathbb{R}^d$, in addition, they are endowed with p-norms: for $0 < p < \infty$

$$\|\boldsymbol{x}\|_{\ell_p^d} := \|\boldsymbol{x}\|_p = \left(\sum_{j=1}^{d} |x_j|^p \right)^{1/p};$$

for $p = \infty$

$$\|\boldsymbol{x}\|_{\ell_\infty^d} := \|\boldsymbol{x}\|_\infty = \max_{j=1,\ldots,d} |x_j|.$$

Note that a different normalization of the ℓ_p^d norm is used in some papers in learning theory (e.g., [51]). For $0 < p \leq \infty$,

$\ell_p := \ell_p^\infty$. We use the shorthand sequence notation $(x_j)_j = (x_1, x_2, \ldots)$.

Given m points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \ell_p^d$, we use the shorthand $\boldsymbol{X}^m = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$.

Suppose $\mathcal{F}$ is a class of functions $f: \mathbb{R}^d \to \mathbb{R}$. The ℓ_∞^d norm *with respect to* $\boldsymbol{X}^m$ of $f \in \mathcal{F}$ is defined as

$$\|f\|_{\ell_\infty^{\boldsymbol{X}^m}} := \max_{i=1,\ldots,m} |f(\boldsymbol{x}_i)|.$$

Likewise,

$$\|f\|_{\ell_p^{\boldsymbol{X}^m}} = \|(f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_m))\|_{\ell_p^m}.$$

Given some set $\mathcal{X}$ with a σ-algebra, a measure μ on $\mathcal{X}$, some $1 \le p < \infty$ and a function $f: \mathcal{X} \to \mathbb{R}$ we define

$$\|f\|_{L_p(\mathcal{X}, \mathbb{R})} := \left(\int |f(x)|^p d\mu(x) \right)^{1/p}$$

if the integral exists and

$$\|f\|_{L_\infty(\mathcal{X}, \mathbb{R})} := \operatorname*{ess\,sup}_{x \in \mathcal{X}} |f(x)|.$$

For $1 \le p \le \infty$, we let

$$L_p(\mathcal{X}, \mathbb{R}) := \{f: \mathcal{X} \to \mathbb{R}: \|f\|_{L_p(\mathcal{X}, \mathbb{R})} < \infty\}.$$

We let $L_p(\mathcal{X}) := L_p(\mathcal{X}, \mathbb{R})$.

If S is a set and d a metric on S, then the *ϵ-covering number of $M \subset S$ with respect to the metric d* denoted $\mathcal{N}(\epsilon, \mathcal{F}, d)$ is the smallest number of elements of an ϵ-cover for $\mathcal{F}$ using the metric d. Given a metric space $E = (S, d)$ we will also write $\mathcal{N}(\epsilon, \mathcal{F}, E)$. The nth *entropy number of a set* $M \subset E$, for $n \in \mathbb{N}$, is

$$\epsilon_n(M) := \inf \{\epsilon > 0: \mathcal{N}(\epsilon, M, E) \le n\}. \tag{7}$$

Let $\mathfrak{L}(E, F)$ be the set of all bounded linear operators T between the normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, i.e., operators such that the image of the (closed) unit ball

$$U_E := \{x \in E: \|x\|_E \le 1\} \tag{8}$$

is bounded. The smallest such bound is called the *operator norm*

$$\|T\| := \sup_{x \in U_E} \|Tx\|_F. \tag{9}$$

The *entropy numbers of an operator* $T \in \mathfrak{L}(E, F)$ are defined as

$$\epsilon_n(T) := \epsilon_n(T(U_E)). \tag{10}$$

Note that $\epsilon_1(T) = \|T\|$, and that $\epsilon_n(T)$ certainly is well-defined for all $n \in \mathbb{N}$ if T is a *compact operator*, i.e., if for any $\epsilon > 0$ there exists a finite cover of $T(U_E)$ with open ϵ balls on $\mathcal{X}$.

The *dyadic entropy numbers of an operator* are defined by

$$e_n(T) := \epsilon_{2^n - 1}(T), \qquad n \in \mathbb{N}. \tag{11}$$

Similarly, the dyadic entropy numbers of a set are defined from its entropy numbers. A very nice introduction to entropy numbers of operators is [13].

In this paper, E and F will always be *Banach spaces*, i.e., complete normed spaces (for instance, ℓ_p^d spaces with $p \ge 1$). In some cases, they will be *Hilbert spaces* H, i.e., Banach spaces endowed with a dot product $\langle \cdot, \cdot \rangle_H$ giving rise to its norm via $\|x\|_H = \sqrt{\langle x, x \rangle_H}$.

By $\log$ and $\ln$, we denote the logarithms to base 2 and e, respectively. By i, we denote the imaginary unit $i = \sqrt{-1}$, k will always be a kernel, and d and m will be the input dimensionality and the number of examples

$$(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_m, y_m) \in \mathbb{R}^d \times \mathbb{R} \tag{12}$$

respectively. We will map the input data into a feature space via a mapping Φ. We let $\tilde{\boldsymbol{x}} := \Phi(\boldsymbol{x})$.

III. Operator Theory Methods for Entropy Numbers

In this section, we briefly explain the new viewpoint utilized in the present paper. With reference to Fig. 1, consider the traditional viewpoint in statistical learning theory. One is given a class of functions $\mathcal{F}$, and the generalization performance attainable using $\mathcal{F}$ is determined via the covering numbers of $\mathcal{F}$. More precisely, for some set $\mathcal{X}$, and $\boldsymbol{x}_i \in \mathcal{X}$ for $i = 1, \ldots, m$, define the *uniform covering numbers* of the function class $\mathcal{F}$ on $\mathcal{X}$ by

$$\mathcal{N}^m(\epsilon, \mathcal{F}) := \sup_{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathcal{X}} \mathcal{N}(\epsilon, \mathcal{F}, \ell_\infty^{\boldsymbol{X}^m}) \tag{13}$$

where $\mathcal{N}(\epsilon, \mathcal{F}, \ell_\infty^{\boldsymbol{X}^m})$ is the ϵ-covering number of $\mathcal{F}$ with respect to $\ell_\infty^{\boldsymbol{X}^m}$. (Recall $\boldsymbol{X}^m = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$.) Many generalization error bounds can be expressed in terms of $\mathcal{N}^m(\epsilon, \mathcal{F})$. An example is given in the following section.

The key novelty in the present work solely concerns the manner in which the covering numbers are computed. Traditionally, appeal has been made to a result such as the so-called Sauer's lemma (originally due to Vapnik and Chervonenkis). In the case of function learning, a generalization called the VC dimension of real-valued functions, or a variation due to Pollard (called the pseudo-dimension), or a scale-sensitive generalization of that (called the fat-shattering dimension) is used to bound the covering numbers. These results reduce the computation of $\mathcal{N}^m(\epsilon, \mathcal{F})$ to the computation of a single "dimension-like" quantity (independent of m). An overview of these various dimensions, some details of their history, and some examples of their computation can be found in [5], [6].

In the present work, we view the class $\mathcal{F}$ as being induced by an operator T_k depending on some kernel function k. Thus, $\mathcal{F}$ is the image of a "base class" $\mathcal{G}$ under T_k. The analogy implicit in the picture is that the quantity that matters is the number of ϵ-distinguishable messages obtainable at the information sink. (Recall the equivalence up to a constant factor of 2 in ϵ of packing and covering numbers [6].) In a typical communications problem, one tries to maximize the number of distinguishable messages (per unit time), in order to maximize the information transmission rate. But from the point of view of the receiver, the decoding job is made easier the *smaller* the number of distinct messages that one needs to be concerned with decoding. (Of course, this lowers the information transmission rate.) The significance of the picture is that the kernel in question is exactly the kernel that is used, for example, in support vector machines.

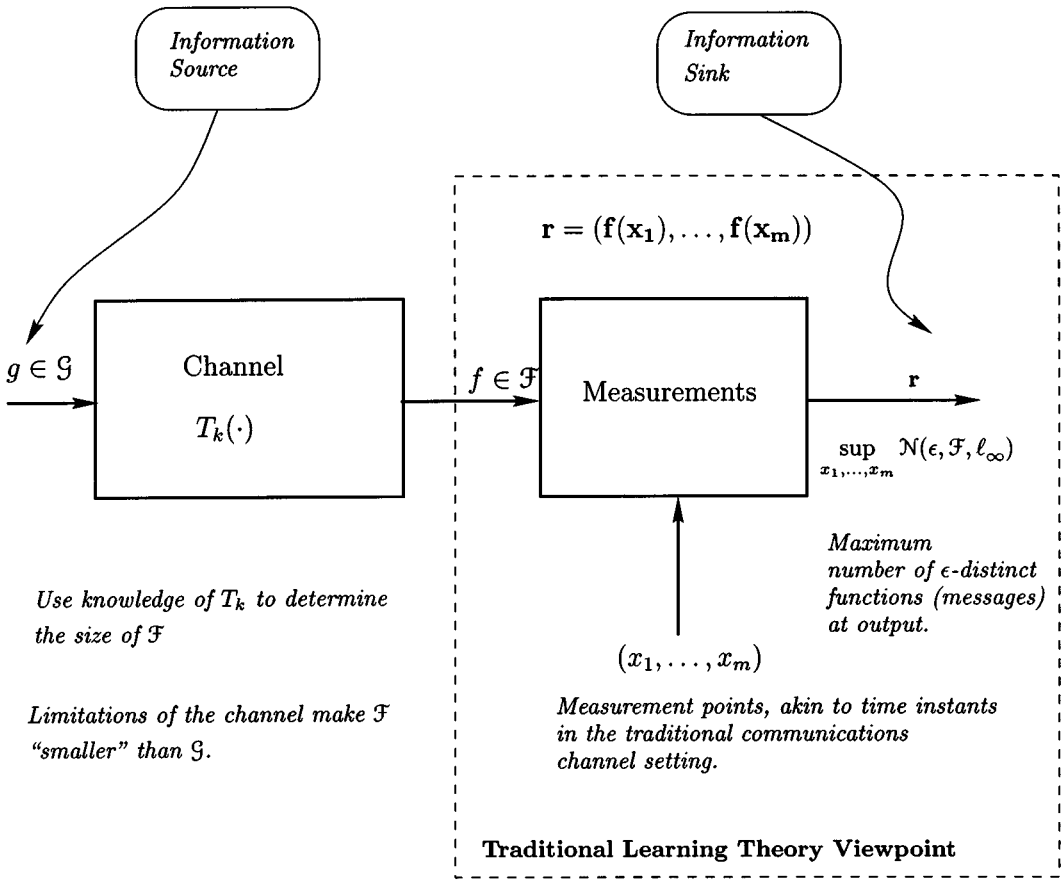


Fig. 1. Schematic picture of the new viewpoint.

As a consequence, the determination of $\mathcal{N}^m(\epsilon, \mathcal{F})$ can be done in terms of properties of the operator T_k. The latter thus plays a constructive role in controlling the complexity of $\mathcal{F}$ and hence the difficulty of the learning task. We believe that the new viewpoint in itself is potentially very valuable, perhaps more so than the specific results in the paper. A further exploitation of the new viewpoint can be found in [62], [61], [49], [47].

We conclude this section with some historical remarks.

The concept of the metric entropy of a set has been around for some time. It seems to have been introduced by Pontriagin and Schnirelmann [37] and was studied in detail by Kolmogorov and others [27] (see also, e.g., [32, Ch. 15]). The use of metric entropy to say something about linear operators was developed independently by several people. Prosser [38] appears to have been the first to make the idea explicit. He determined the effect of an operator's spectrum on its entropy numbers. In particular, he proved a number of results concerning the asymptotic rate of decrease of the entropy numbers in terms of the asymptotic behavior of the eigenvalues. A similar result is actually implicit in Shannon's famous paper [42, Sec. 22], where he considered the effect of different convolution operators on the entropy of an ensemble. Prosser's paper [38] led to a handful of papers (see, e.g., [39], [22], [3], [29]) which studied various convolutional operators. A connection between Prosser's ϵ-entropy of an operator and Kolmogorov's ϵ-entropy of a stochastic process was shown in [2]. Independently, another group of mathematicians including Carl and Stephani [13] studied covering numbers [52] and later entropy numbers [36] in the context of operator ideals. (They were unaware of Prosser's work—see, e.g., [11, p. 136].)

Connections between the local theory of Banach spaces and uniform convergence of empirical means has been noted before (e.g., [35]). More recently, Gurvits [21] has obtained a result relating the Rademacher type of a Banach space to the fat-shattering dimension of linear functionals on that space and hence via the key result in [4] to the covering numbers of the induced class. We will make further remarks concerning the relationship between Gurvits' approach and ours in [60]; for now, let us just note that the equivalence of the type of an operator (or of the space it maps to), and the rate of decay of its entropy numbers has been (independently) shown by Kolchinskiy[25], [26] and Defant and Junge [15], [23]. Note that the exact formulation of their results differs. Kolchinskiy was motivated by probabilistic problems not unlike ours.

IV. Generalization Bounds via Uniform Convergence

The generalization performance of learning machines can be bounded via uniform convergence results as in [57], [56]. A recent review can be found in [5]; see also [30]. The key thing about these results is the role of the covering numbers of the hypothesis class—the focus of the present paper. Results for both classification and regression are now known. For the sake of concreteness, we quote below a result suitable for regression

which was proved in [4]. For results on classifier performance in terms of covering numbers see [8]. Let

$$P_m(f) := \left(\frac{1}{m}\right)\sum_{i=1}^{m} f(\boldsymbol{x}_i)$$

denote the *empirical mean* of f on the sample $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.

We make use of the following due to Alon, Ben-David, Cesa-Bianchi, and Haussler [4].

Lemma 1: Let $\mathcal{F}$ be a class of functions from $\mathcal{X}$ into $[0, 1]$ and let P be a distribution over $\mathcal{X}$. Then, for all $\epsilon > 0$ and all $m \geq \frac{2}{\epsilon^2}$

$$\Pr\left\{\sup_{f\in\mathcal{F}} |P_m(f) - P(f)| > \epsilon\right\} \leq 12m \cdot \boldsymbol{E}\left[\mathcal{N}\left(\frac{\epsilon}{6}, \mathcal{F}, \ell_\infty^{\overline{\boldsymbol{X}}^{2m}}\right)\right] e^{-\epsilon^2 m/36} \quad (14)$$

where Pr denotes the probability with respect to the sample $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ drawn independent and identically distributred (i.i.d.) from P, and $\boldsymbol{E}$ the expectation with respect to a second sample $\overline{\boldsymbol{X}}^{2m} = (\overline{\boldsymbol{x}}_1, \ldots, \overline{\boldsymbol{x}}_{2m})$ also drawn i.i.d. from P.

In order to use this lemma, one usually makes use of the fact that for any P

$$\boldsymbol{E}\left[\mathcal{N}\left(\epsilon, \mathcal{F}, \ell_\infty^{\tilde{\boldsymbol{X}}^m}\right)\right] \leq \mathcal{N}^m(\epsilon, \mathcal{F}). \quad (15)$$

The above result can be used to give a generalization error result by applying it to the loss-function-induced class. The following lemma, which is an improved version of [9, Lemma 17], is useful in this regard (a similar result appears in [6]).

Lemma 2: Let $\mathcal{F}$ be a set of functions from $\mathcal{X}$ to $[a, b]$ with $a < b$, $a, b \in \mathbb{R}$, and $l: \mathbb{R} \to [0, \infty)$ a loss function. Let

$$\begin{aligned} \boldsymbol{X}^m &= (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \\ \boldsymbol{z}_i &:= (\boldsymbol{x}_i, y_i) \\ \boldsymbol{Z}^m &= (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m) \\ l_f|_{\boldsymbol{z}_j} &:= l(f(\boldsymbol{x}_j) - y_j) \\ l_f|_{\boldsymbol{Z}^m} &:= (l_f|_{\boldsymbol{z}_j})_{j=1}^m \\ l_{\mathcal{F}}|_{\boldsymbol{Z}^m} &:= \{l_f|_{\boldsymbol{Z}^m} : f \in \mathcal{F}\} \end{aligned}$$

and

$$\mathcal{N}(\epsilon, l|_{\boldsymbol{Z}^m}) := \mathcal{N}(\epsilon, l_{\mathcal{F}}|_{\boldsymbol{Z}^m}, \ell_\infty^{\boldsymbol{Z}^m}).$$

Then the following two statements hold.

1) Suppose l satisfies the Lipschitz condition

$$l(\xi) - l(\xi') \leq C|\xi - \xi'|, \quad \text{for all } \xi, \xi' \in [a-b, b-a]. \quad (16)$$

Then for all $\epsilon > 0$

$$\max_{\boldsymbol{Z}^m \in (\mathcal{X}\times[a,b])^m} \mathcal{N}(\epsilon, l|_{\boldsymbol{Z}^m}) \leq \max_{\boldsymbol{X}^m \in \mathcal{X}^m} \mathcal{N}\left(\frac{\epsilon}{C}, \mathcal{F}|_{\boldsymbol{X}^m}, \ell_\infty^m\right) \quad (17)$$

and

$$\max_{\boldsymbol{Z}^m \in (\mathcal{X}\times[a,b])^m} \mathcal{N}(\epsilon, l|_{\boldsymbol{Z}^m}) \leq \max_{\boldsymbol{X}^m \in \mathcal{X}^m} \mathcal{N}\left(\frac{\epsilon m}{C}, \mathcal{F}|_{\boldsymbol{X}^m}, \ell_1^m\right). \quad (18)$$

2) Suppose that for some $C, \tilde{C} > 0$, l satisfies the "approximate Lipschitz condition"

$$l(\xi) - l(\xi') \leq \max(C|\xi - \xi'|, \tilde{C}), \quad \text{for all } \xi, \xi' \in [a-b, b-a] \quad (19)$$

then for all $\epsilon > \tilde{C}/C$

$$\max_{\boldsymbol{Z}^m \in (\mathcal{X}\times[a,b])^m} \mathcal{N}(\epsilon, l|_{\boldsymbol{Z}^m}) \leq \max_{\boldsymbol{X}^m \in \mathcal{X}^m} \mathcal{N}\left(\frac{\epsilon}{C}, \mathcal{F}|_{\boldsymbol{X}^m}, \ell_\infty^m\right). \quad (20)$$

Proof: We show that, for any sequence $\boldsymbol{Z}^m$ of $(\boldsymbol{x}, y)$ pairs in $\mathcal{X} \times [a, b]$ and any functions f and g, if the restrictions of f and g to $\boldsymbol{X}^m$ are close, then the restrictions of l_f and l_g to $\boldsymbol{Z}^m$ are close. Thus, given a cover of $\mathcal{F}|_{\boldsymbol{X}^m}$ we can construct a cover of $l_{\mathcal{F}}|_{\boldsymbol{Z}^m}$ that is no bigger. For case 1) we get

$$\begin{aligned} &\frac{1}{m}\left|\sum_{j=1}^{m} l(g(\boldsymbol{x}_j) - y_j) - l(f(\boldsymbol{x}_j) - y_j)\right| \\ &\quad \leq \frac{1}{m}\sum_{j=1}^{m} |l(g(\boldsymbol{x}_j) - y_j) - l(f(\boldsymbol{x}_j) - y_j)| \\ &\quad \leq \frac{1}{m}\sum_{j=1}^{m} C|g(\boldsymbol{x}_j) - f(\boldsymbol{x}_j)| \\ &\quad = \frac{C}{m}\|g(\tilde{\boldsymbol{X}}^m) - f(\tilde{\boldsymbol{X}}^m)\|_{\ell_1^m} \\ &\quad \leq C\|g(\tilde{\boldsymbol{X}}^m) - f(\tilde{\boldsymbol{X}}^m)\|_{\ell_\infty^m}. \end{aligned}$$

In the second case we proceed similarly

$$\begin{aligned} &\frac{1}{m}\left|\sum_{j=1}^{m} l(g(\boldsymbol{x}_j) - y_j) - l(f(\boldsymbol{x}_j) - y_j)\right| \\ &\quad \leq \frac{C}{m}\sum_{j=1}^{m} \max(|g(\boldsymbol{x}_j) - f(\boldsymbol{x}_j)|, \tilde{C}/C) \\ &\quad \leq C\epsilon, \qquad \text{for } \epsilon \geq \tilde{C}/C. \end{aligned}$$

□

The second case can be useful when the exact form of the cost function is not known, happens to be discontinuous, or is badly behaved in some other way. Applying the result above to polynomial loss leads to the following corollary.

Corollary 3: Let the assumptions be as above in Lemma 2. Then for loss functions of type

$$l(\eta) = \frac{1}{p}\eta^p, \qquad \text{with } p > 1 \quad (21)$$

we have $C = (b-a)^{(p-1)}$, in particular $C = (b-a)$ for $p = 2$ and, therefore,

$$\max_{\boldsymbol{z}\in(\mathcal{X}\times[a,b])^m} \mathcal{N}(\epsilon, l|_{\boldsymbol{z}}) \leq \max_{\boldsymbol{x}\in\mathcal{X}^m} \mathcal{N}\left(\frac{\epsilon}{(b-a)^{p-1}}, \mathcal{F}|_{\boldsymbol{x}}\right). \quad (22)$$

One can readily combine the uniform convergence results with the above results to get overall bounds on generalization performance. We do not explicitly do this here since the particular uniform convergence result needed depends on the exact setup

of the learning problem. A typical uniform convergence result takes the form

$$P^m\left\{\sup_f |R_{\rm emp}(f)-R(f)|>\epsilon\right\} \le c_1(m)\mathcal{N}^m(\epsilon,\mathcal{F})e^{-\epsilon^\beta m/c_2} \quad (23)$$

where $R_{\rm emp}(f)$ is the empirical risk and $R(f)$ the expected risk of $f\in\mathcal{F}$ (see, e.g., [6], [54] Even the exponent in (23) depends on the setting:. In regression, β can be set to 1, however, in agnostic learning [24] in general $\beta=2$, except if the class is convex in which case it can be set to 1 [31]. Since our primary interest is in determining $\mathcal{N}^m(\epsilon,\mathcal{F})$ we will not try to summarize the large body of results now available on uniform convergence and generalization error.

These generalization bounds are typically used by setting the right-hand side equal to δ and solving for $m=m(\epsilon,\delta)$ (which is called the sample complexity). Another way to use these results is as a learning curve bound $\overline{\epsilon}(\delta,m)$ where

$$P^m\left\{\sup_{f\in\mathcal{F}} |R_{\rm emp}(f)-R(f)|>\overline{\epsilon}(\delta,m)\right\}\le\delta.$$

We note here that the determination of $\overline{\epsilon}(\delta,m)$ is quite convenient in terms of e_n, the dyadic entropy number associated with the covering number $\mathcal{N}^m(\epsilon,\mathcal{F})$ in (23). Setting the right-hand side of (23) equal to δ, we have

$$\begin{aligned}\delta &= c_1(m)\mathcal{N}^m(\epsilon,\mathcal{F})e^{-\epsilon^\beta m/c_2}\\ \Rightarrow\quad \log\left(\frac{\delta}{c_1(m)}\right)+\frac{\epsilon^\beta m}{c_2\ln 2} &= \log\mathcal{N}^m(\epsilon,\mathcal{F})\\ \Rightarrow\quad e_{\lceil\log\left(\frac{\delta}{c_1(m)}\right)+\frac{\epsilon^\beta m}{c_2\ln 2}+1\rceil} &\le \epsilon. \end{aligned}\quad (24)$$

Thus, $\overline{\epsilon}(\delta,m)=\min\{\epsilon\colon$ (24) holds$\}$. Hence, the use of ϵ_n or e_n (which will arise naturally from our techniques) is in fact a convenient thing to do for finding learning curves.

V. Entropy Numbers for Kernel Machines

In the following, we will mainly consider machines where the mapping into feature space is defined by Mercer kernels $k(\boldsymbol{x},\boldsymbol{y})$ as they are easier to deal with using functional analytic methods. (More general kernels are considered in [47].) Such machines have become very popular due to the success of SV machines.

A. Mercer's Theorem, Feature Spaces, and Scaling

Our goal is to make statements about the shape of the image of the input space $\mathcal{X}$ under the feature map $\Phi(\cdot)$. We will make use of Mercer's theorem. The version stated below is a special case of the theorem proven in [28, p. 145]. In the following we will assume $(\mathcal{X},\mu)$ to be a finite measure space, i.e., $\mu(\mathcal{X})<\infty$.

Theorem 4 (Mercer): Suppose $k\in L_\infty(\mathcal{X}\times\mathcal{X})$ is a symmetric kernel (that is, $k(x,x')=k(x',x)$) such that the integral operator $T_k\colon L_2(\mathcal{X})\to L_2(\mathcal{X})$

$$T_kf(\cdot):=\int_{\mathcal{X}} k(\cdot,\boldsymbol{y})f(\boldsymbol{y})\,d\mu(\boldsymbol{y}) \quad (25)$$

is positive. Let $\psi_j\in L_2(\mathcal{X})$ be the eigenfunction of T_k associated with the eigenvalue $\lambda_j\neq 0$ and normalized by $\|\psi_j\|_{L_2}=1$. Suppose ψ_j is continuous for all $j\in\mathbb{N}$. Then

1) $(\lambda_j(T))_j\in\ell_1$ for $j=1,2,\ldots$;
2) $\psi_j\in L_\infty(\mathcal{X})$ and $\sup_j\|\psi_j\|_{L_\infty}<\infty$;
3) $k(\boldsymbol{x},\boldsymbol{y})=\sum_{j\in\mathbb{N}}\lambda_j\psi_j(\boldsymbol{x})\psi_j(\boldsymbol{y})$ holds for all $(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}\times\mathcal{X}$;

where the series converges absolutely and uniformly for all $(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}\times\mathcal{X}$.

We will call a kernel satisfying the conditions of this theorem a *Mercer kernel*. Note that if $\mathcal{X}$ is compact and k is continuous, then ψ_j is continuous (cf., e.g., [7, p. 270]). Alternatively, if k is translation-invariant, then ψ_j are scaled cosine functions and thus continuous. Thus, the assumption that ψ_j are continuous is not very restrictive.

From statement 2) of Mercer's theorem there exists some constant $C_k\in\mathbb{R}^+$ depending on $k(\cdot,\cdot)$ such that

$$|\psi_j(\boldsymbol{x})|\le C_k,\qquad \text{for all } j\in\mathbb{N}\text{ and }\boldsymbol{x}\in\mathcal{X}. \quad (26)$$

Moreover, from statement 3) it follows that $k(\boldsymbol{x},\boldsymbol{y})$ corresponds to a dot product in ℓ_2, i.e., $k(\boldsymbol{x},\boldsymbol{y})=\langle\Phi(\boldsymbol{x}),\Phi(\boldsymbol{y})\rangle_{\ell_2}$ with

$$\begin{aligned}\Phi\colon\ \mathcal{X} &\to \ell_2\\ \Phi\colon\ \boldsymbol{x} &\mapsto (\phi_j(\boldsymbol{x}))_j:=\left(\sqrt{\lambda_j}\,\psi_j(\boldsymbol{x})\right)_j\end{aligned}\quad (27)$$

for all $\boldsymbol{x}\in\mathcal{X}$. In the following, we will (without loss of generality) assume the sequence of $(\lambda_j)_j$ is sorted in nonincreasing order. From the argument above, one can see that $\Phi(\mathcal{X})$ lives not only in ℓ_2 but in an axis parallel parallelepiped with lengths $2C_k\sqrt{\lambda_j}$.

We remark that the measure μ need have nothing to do with the distribution of examples. In particular, we may consider any of the following kernels in our bounds.

Lemma 5 (Equivalent Kernels): Denote by $\mathcal{X}$ a compact set and by $k\colon\mathcal{X}^2\to\mathbb{R}$ a Mercer kernel. Then ,for any $\mathcal{X}'$ and surjective map $\chi\colon\mathcal{X}'\to\mathcal{X}$, the kernel $k'(x,x'):=k(\chi(x),\chi(x'))$ also satisfies Mercer's condition and, moreover, the eigenvalues λ_i' and the coefficient $C_{k'}$ of the integral operator

$$T_{k'}f(x):=\int_{\mathcal{X}'}k'(x,x')f(x')\,dx' \quad (28)$$

can be used equivalently in any application of k.

This means in particular that we could construct diffeomorphisms $\chi\colon\mathcal{X}\to\mathcal{X}$ and look for the function χ such that the eigenvalues λ_i' and $C_{k'}$ are as small as possible.

Proof: The first part of the claim, namely, that k' also satisfies Mercer's condition, follows immediately from the construction of k'. For the second claim, note that due to the fact that χ is surjective for any distribution $p(x)$ on $\mathcal{X}$ there must exist an equivalent distribution $p'(x)$ on $\mathcal{X}'$. Thus, we can always consider the problem as being one on $\mathcal{X}'$ from the start. However, since $\mathcal{X}'$ and χ' were chosen arbitrarily we can optimize over them. □

Lemma 5 shows that the specific bounds we obtain *will* depend on μ since that will affect the ψ_j, C_k, and the $(\lambda_j)_j$. The question of the optimal μ to use and how it may be chosen if one knows P (the distribution from which the $\boldsymbol{x}_j$ are drawn) is not considered here. In all cases considered in this paper, we will in fact take μ to be the Lebesgue measure.

It will be useful to consider maps that map $\Phi(\mathcal{X})$ into balls of some radius R centered at the origin. The following proposition

shows that the class of all these maps is determined by elements of ℓ_2 and the sequence of eigenvalues $(\lambda_j)_j$.

Proposition 6 (Mapping $\Phi(\mathcal{X})$ into ℓ_2): Let S be the diagonal map

$$S\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$$
$$S\colon (x_j)_j \mapsto S(x_j)_j = (s_j x_j)_j \qquad \text{with } s_j \in \mathbb{R}. \tag{29}$$

Then S maps $\Phi(\mathcal{X})$ into a ball of finite radius R_S centered at the origin if and only if $(s_j\sqrt{\lambda_j}\, s_j)_j \in \ell_2$.

Proof:

($\Leftarrow$) Suppose $(s_j\sqrt{\lambda_j})_j \in \ell_2$ and let

$$R_S^2 := C_k^2 \|(s_j\sqrt{\lambda_j})_j\|_{\ell_2}^2 < \infty.$$

For any $\boldsymbol{x} \in \mathcal{X}$

$$\|S\Phi(\boldsymbol{x})\|_{\ell_2}^2 = \sum_{j\in\mathbb{N}} s_j^2\lambda_j|\psi_j(\boldsymbol{x})|^2 \le \sum_{j\in\mathbb{N}} s_j^2\lambda_j C_k^2 = R_S^2. \tag{30}$$

Hence $S\Phi(\mathcal{X}) \subseteq \ell_2$.

($\Rightarrow$) Suppose $(s_j\sqrt{\lambda_j})_j$ is not in ℓ_2. Hence the sequence $(A_n)_n$ with

$$A_n := \sum_{j=1}^{n} s_j^2\lambda_j$$

is unbounded. Now define

$$a_n(\boldsymbol{x}) := \sum_{j=1}^{n} s_j^2\lambda_j|\psi_j(\boldsymbol{x})|^2. \tag{31}$$

Then $\|a_n(\cdot)\|_{L_1(\mathcal{X})} = A_n$ due to the normalization condition on ψ_j. However, as $\mu(\mathcal{X}) < \infty$ there exists a set $\tilde{\mathcal{X}}$ of nonzero measure such that

$$a_n(\boldsymbol{x}) \ge \frac{A_n}{\mu(\mathcal{X})}, \qquad \text{for all } \boldsymbol{x} \in \tilde{\mathcal{X}}. \tag{32}$$

Combining the left-hand side of (30) with (31) we obtain

$$\|S\Phi(\boldsymbol{x})\|_{\ell_2}^2 \ge a_n(\boldsymbol{x}), \qquad \text{for all } n \in \mathbb{N} \text{ and all } \boldsymbol{x}.$$

Since $a_n(\boldsymbol{x})$ is unbounded for a set $\tilde{\mathcal{X}}$ with nonzero measure in $\mathcal{X}$, we can see that $S\Phi(\mathcal{X}) \not\subset \ell_2$. □

Once we know that $\Phi(\mathcal{X})$ is contained in the parallelepiped described above we can use this result to construct a mapping $\hat{A}$ from the unit ball in ℓ_2 to an ellipsoid $\mathcal{E}$ such that $\Phi(\mathcal{X}) \subset \mathcal{E}$ as in the following diagram (where we have slightly abused the traditional notational convention).

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\Phi} & \Phi(\mathcal{X}) \subset \ell_2 & \xrightarrow{\hat{A}^{-1}} & U_{\ell_2} \subset \ell_2 \\ & & \cap & \nearrow_{\hat{A}} & \\ & & \ell_2 \supset \mathcal{E} & & \end{array} \tag{33}$$

The operator $\hat{A}$ will be useful for computing the entropy numbers of concatenations of operators. (Knowing the inverse will allow us to compute the forward operator, and that can be used to bound the covering numbers of the class of functions, as shown in the next subsection.) We thus seek an operator $\hat{A}\colon \ell_2 \to \ell_2$ such that

$$\hat{A}^{-1}\Phi(\mathcal{X}) \subset U_{\ell_2}. \tag{34}$$

This means that $\mathcal{E} := AU_{\ell_2}$ will be such that $\Phi(\mathcal{X}) \subset \mathcal{E}$. The latter can be ensured by constructing $\hat{A}$ such that

$$\hat{A}\colon (x_j)_j \mapsto (R_{\hat{A}} \cdot a_j \cdot x_j)_j, \qquad \text{with } R_{\hat{A}},\, a_j \in \mathbb{R}^+ \tag{35}$$

where C_k and a_j are chosen with respect to a specific kernel and where $R_{\hat{A}} := C_k\|(\sqrt{\lambda_j}/a_j)_j\|_{\ell_2}$. From Proposition 6, it follows that all those operators $\hat{A}$ for which $R_{\hat{A}} < \infty$ will satisfy (34). We call such scaling (inverse) operators *admissible.*

B. Entropy Numbers

The next step is to compute the entropy numbers of the operator $\hat{A}$ and use this to obtain bounds on the entropy numbers for kernel machines like SV machines. We will make use of the following theorem due to Gordon, König, and Schütt [17, p. 226] (stated in the present form in [13, p. 17]).

Theorem 7: Let $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_j \ge \cdots \ge 0$ be a nonincreasing sequence of nonnegative numbers and let

$$D\boldsymbol{x} = (\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_j x_j, \ldots) \tag{36}$$

for $\boldsymbol{x} = (x_1, x_2, \ldots, x_j, \ldots) \in \ell_p$ be the diagonal operator from ℓ_p into itself, generated by the sequence $(\sigma_j)_j$, where $1 \le p \le \infty$. Then for all $n \in \mathbb{N}$

$$\sup_{j\in\mathbb{N}} n^{-\frac{1}{j}}(\sigma_1\sigma_2\cdots\sigma_j)^{\frac{1}{j}} \le \epsilon_n(D) \le 6 \sup_{j\in\mathbb{N}} n^{-\frac{1}{j}}(\sigma_1\sigma_2\cdots\sigma_j)^{\frac{1}{j}}. \tag{37}$$

We can exploit the freedom in choosing $\hat{A}$ to minimize an entropy number as the following corollary shows. This will be a key ingredient of the calculation of the covering numbers for SV classes, as shown below.

Proposition 8 (Scaling Operators): Let $k\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be a Mercer kernel with eigenvalues $(\lambda_s)_s$. Choose $a_j > 0$ for $j \in \mathbb{N}$ such that $(\sqrt{\lambda_s}/a_s)_s \in \ell_2$, and define

$$A\colon (x_j)_j \mapsto (R_A a_j x_j)_j \tag{38}$$

with $R_A := C_k\|(\sqrt{\lambda_j}/a_j)_j\|_{\ell_2}$. Then

$$\epsilon_n(A\colon \ell_2 \to \ell_2) \le \sup_{j\in\mathbb{N}} 6C_k \left\|\left(\sqrt{\lambda_s}/a_s\right)_s\right\|_{\ell_2} \left(\frac{a_1\cdots a_j}{n}\right)^{\frac{1}{j}}. \tag{39}$$

This result follows immediately by identifying D and A. We can optimize (39) over all possible choices of A to obtain the following proposition. (It turns out that the infimum is in fact attainable [20] when k is a Mercer kernel thus justifying writing the inequality as we do. That is, we can minimize the right-hand side of (39).)

Proposition 9: There exists an $\hat{A}$ defined by (35) that satisfies

$$\epsilon_n(\hat{A}) \le \inf_{(a_s)_s:\left(\sqrt{\lambda_s}/a_s\right)_s\in\ell_2} \sup_{j\in\mathbb{N}} 6C_k \left\|\left(\sqrt{\lambda_s}/a_s\right)_s\right\|_{\ell_2} \cdot n^{-\frac{1}{j}}(a_1a_2\cdots a_j)^{\frac{1}{j}}. \tag{40}$$

As already described in Section I, the hypothesis that an SV machine generates can be expressed as $\langle \boldsymbol{w}, \tilde{\boldsymbol{x}}\rangle + b$ where both $\boldsymbol{w}$ and $\tilde{\boldsymbol{x}}$ are defined in the feature space $\mathcal{S} = \operatorname{span}(\Phi(\mathcal{X}))$

and $b \in \mathbb{R}$. The kernel trick, as introduced by [1], was then successfully employed in [10] and [14] to extend the optimal margin hyperplane classifier to what is now known as the SV machine. We deal with the "$+b$" term in Section IX; for now we consider the class

$$\mathcal{F}_\Lambda := \{f_{\boldsymbol{w}}: \boldsymbol{x} \mapsto \langle \boldsymbol{w}, \boldsymbol{x}\rangle: \boldsymbol{x} \in \mathcal{S}, \|\boldsymbol{w}\| \le \Lambda\} \subseteq \mathbb{R}^{\mathcal{S}}.$$

Note that $\mathcal{F}_\Lambda$ depends implicitly on k since $\mathcal{S}$ does.

We seek the ℓ_∞^m covering numbers for the class $\mathcal{F}_\Lambda$ induced by the kernel in terms of the parameter Λ which is the inverse of the size of the margin in feature space, or equivalently, the size of the weight vector in feature space as defined by the dot product in $\mathcal{S}$ (see [55], and [53] for details). In the following, we will call such hypothesis classes with length constraint on the weight vectors in feature space *SV classes*. Let T be the operator $T = S_{\tilde{\boldsymbol{X}}^m}\Lambda$ where $\Lambda \in \mathbb{R}^+$ and the operator $S_{\tilde{\boldsymbol{X}}^m}$ is defined by

$$\begin{aligned} &S_{\tilde{\boldsymbol{X}}^m}: \ell_2 \to \ell_\infty^m \\ &S_{\tilde{\boldsymbol{X}}^m}: \boldsymbol{w} \mapsto (\langle \tilde{\boldsymbol{x}}_1, \boldsymbol{w}\rangle, \ldots, \langle \tilde{\boldsymbol{x}}_m, \boldsymbol{w}\rangle) \end{aligned} \tag{41}$$

with $\tilde{\boldsymbol{x}}_j \in \Phi(\mathcal{X})$ for all j. The following theorem is useful when computing entropy numbers in terms of T and A. It is originally due to Maurey, and was extended by Carl [12] and is given in almost the form below by Carl and Stephani [13, p. 246].

Theorem 10 (Maurey): Let $S \in \mathfrak{L}(H, \ell_\infty^m)$ where H is a Hilbert space. Then, there exists a constant $c > 0$ such that for all $n, m \in \mathbb{N}$

$$e_n(S) \le c\|S\| \left(n^{-1} \log\left(1 + \frac{m}{n}\right)\right)^{1/2}. \tag{42}$$

(Carl and Stephani state an additional condition, namely, that $n \le m$. It turns out [62] that for $n > m$, and even tighter bound holds, and so it is not incorrect to state it as above. It should be added that this tighter bound is of little value in learning theory applications: it corresponds to determining the ϵ-covering number for extremely small ϵ for which $\log \mathcal{N}^m(\epsilon, \mathcal{F}) > m$.)

An alternative proof of this result (given in [62]) provides a small explicit value for the constant: $c \le 103$. However, there is reason to believe that c should be 1.86, the constant obtainable for identity maps from ℓ_2^m into ℓ_∞^m.

The restatement of Theorem 10 in terms of $\epsilon_{2^{n-1}} = e_n$ will be useful in the following. Under the assumptions above we have

$$\epsilon_n(S) \le c\|S\| \left((\log n + 1)^{-1} \log\left(1 + \frac{m}{\log n + 1}\right)\right)^{1/2}. \tag{43}$$

Now we can combine the bounds on entropy numbers of A and $S_{\tilde{\boldsymbol{X}}^m}$ to obtain bounds for SV classes. First we need the following lemma from [13, p. 11].

Lemma 11 (Carl and Stephani): Let E, F, G be Banach spaces, $R \in \mathfrak{L}(F, G)$, and $S \in \mathfrak{L}(E, F)$. Then, for $n, t \in \mathbb{N}$

$$\epsilon_{nt}(RS) \le \epsilon_n(R)\epsilon_t(S) \tag{44}$$

$$\epsilon_n(RS) \le \epsilon_n(R)\|S\| \tag{45}$$

$$\epsilon_n(RS) \le \epsilon_n(S)\|R\|. \tag{46}$$

Note that the latter two inequalities follow directly from (44) and the fact that $\epsilon_1(R) = \|R\|$ for all $R \in \mathfrak{L}(F, G)$.

Proposition 12 (Bounds for SV Classes): Let k be a Mercer kernel, let Φ be induced via (27), and let $T := S_{\tilde{\boldsymbol{X}}^m}\Lambda$, where $S_{\tilde{\boldsymbol{X}}^m}$ is given by (41) and $\Lambda \in \mathbb{R}^+$. Let A be defined as in Proposition 9 and suppose $\tilde{\boldsymbol{x}}_j = \Phi(\boldsymbol{x}_j)$ for $j = 1, \ldots, m$. Then, the entropy numbers of T satisfy the following inequalities:

$$\epsilon_n(T) \le c\|A\|\Lambda \log^{-1/2} n \log^{1/2}\left(1 + \frac{m}{\log n}\right) \tag{47}$$

$$\epsilon_n(T) \le 6\Lambda\epsilon_n(A) \tag{48}$$

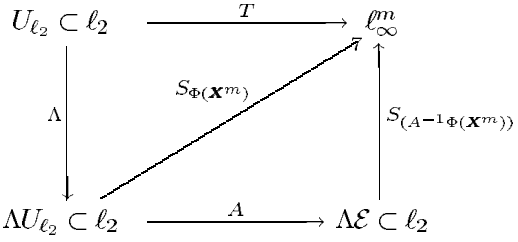
$$\epsilon_{nt}(T) \le 6c\Lambda \log^{-1/2} n \log^{1/2}\left(1 + \frac{m}{\log n}\right)\epsilon_t(A) \tag{49}$$

where c is defined as in Theorem 10.

This result gives several options for bounding $\epsilon_n(T)$. We shall see in examples later that the best inequality to use depends on the rate of decay of the eigenvalues of k. The result gives effective bounds on $\mathcal{N}^m(\epsilon, \mathcal{F}_\Lambda)$ since

$$\epsilon_n(T: \ell_2 \to \ell_\infty^m) \le \epsilon_0 \Rightarrow \mathcal{N}^m(\epsilon_0, \mathcal{F}_\Lambda) \le n.$$

Proof: We will use the following factorization of T to upper-bound $\epsilon_n(T)$.



(50)

The top arrow in the diagram follows from the definition of T. The fact that remainder commutes stems from the fact that since A is diagonal, it is self-adjoint and so for any $\tilde{\boldsymbol{x}} \in \mathcal{S}$

$$\langle \boldsymbol{w}, \tilde{\boldsymbol{x}}\rangle = \langle \boldsymbol{w}, AA^{-1}\tilde{\boldsymbol{x}}\rangle = \langle A\boldsymbol{w}, A^{-1}\tilde{\boldsymbol{x}}\rangle. \tag{51}$$

Instead of computing the entropy number of $T = S_{\tilde{\boldsymbol{X}}^m}\Lambda$ directly, which is difficult or wasteful, as the bound on $S_{\tilde{\boldsymbol{X}}^m}$ does not take into account that $\tilde{\boldsymbol{x}} \in \mathcal{E}$ but just makes the assumption of $\tilde{\boldsymbol{x}} \in \rho U_{\ell_2}$ for some $\rho > 0$, we will represent T as $S_{(A^{-1}\tilde{\boldsymbol{X}}^m)}A\Lambda$. This is more efficient as we constructed A such that $A^{-1}\Phi(\mathcal{X}) \subseteq U_{\ell_2}$ filling a larger proportion of it than $\frac{1}{\rho}\Phi(\mathcal{X})$ does.

By construction of A and due to the Cauchy–Schwarz inequality we have $\|S_{A^{-1}\tilde{\boldsymbol{X}}^m}\| \le 1$. Thus applying Lemma 11 to the factorization of T and using Theorem 10 proves the theorem. □

As we shall see in Section VII, one can give asymptotic rates of decay for $\epsilon_n(A)$. (In fact, we give nonasymptotic results with explicitly evaluable constants.) It is thus of some interest to give overall asymptotic rates of decay of $\epsilon_n(T)$ in terms of the order of $\epsilon_n(A)$. (By "asymptotic" here we mean asymptotic in n; this corresponds to asking how $\mathcal{N}^m(\epsilon, \mathcal{F})$ scales as $\epsilon \to 0$ for fixed m.)

Lemma 13 (Rate Bounds on ϵ_n): Let k be a Mercer kernel and suppose A is the scaling operator associated with it as defined by (38).

1) If $\epsilon_n(A)=O(\log^{-\alpha} n)$ for some $\alpha>0$ then for fixed m
$$\epsilon_n(T) = O(\log^{-(\alpha+1/2)} n). \tag{52}$$
2) If $\log \epsilon_n(A) = O(\log^{-\beta} n)$ for some $\beta > 0$ then for fixed m
$$\log \epsilon_n(T) = O(\log^{-\beta} n). \tag{53}$$

This Lemma shows that in the first case, Maurey's result (Theorem 10) allows an improvement in the exponent of the entropy number of T, whereas in the second, it affords none (since the entropy numbers decay so fast anyway). The Maurey result may still help in that case though for nonasymptotic n.

Proof: From theorem 10 we know that
$$\epsilon_n(S) = O(\log^{-1/2} n).$$
Now use (49), ignoring constants and assuming m is fixed, splitting the index n in the following way:
$$n = n^{\tau} n^{(1-\tau)}, \qquad \text{with } \tau \in (0, 1). \tag{54}$$
For the first case this yields
$$\begin{aligned}\epsilon_n(T) &\le \epsilon_{n^{\tau}}(S)\epsilon_{n^{1-\tau}}(A)\\ &= (\log^{-1/2} n^{\tau})(\log^{-\alpha} n^{1-\tau})\\ &= \tau^{-1/2}(1-\tau)^{-\alpha}(\log n)^{-1/2-\alpha}\\ &= O(\log^{-(1/2+\alpha)} n).\end{aligned}$$
In the second case we have
$$\begin{aligned}&\log \epsilon_n(T)\\ &\quad= \log(\tau^{-1/2}O(\log^{-1/2} n)) + (1-\tau)^{-\beta}O(\log^{-\beta} n)\\ &\quad= O(\log^{-\beta} n).\end{aligned} \tag{55}$$
□

In a nutshell, we can always obtain rates of convergence better than those due to Maurey's theorem because we are not dealing with *arbitrary* mappings into infinite-dimensional spaces. In fact, for logarithmic dependency of $\epsilon_n(T)$ on n, the effect of the kernel is so strong that it completely dominates the $1/\sqrt{n}$ behavior for arbitrary Hilbert spaces. An example of such a kernel is $k(x, y) = \exp(-(x-y)^2)$; see Proposition 17 and also Section VI for the discretization question.

VI. Discrete Spectra of Convolution Operators

The results presented above show that if one knows the eigenvalue sequence $(\lambda_i)_i$ of a compact operator, one can bound its entropy numbers. While it is always possible to assume that the *data* fed into an SV machine have bounded support, the same cannot be said of the kernel $k(\cdot, \cdot)$; a commonly used kernel is $k(x, y) = \exp(-(x-y)^2)$ which has noncompact support. The induced integral operator
$$(T_k f)(x) = \int_{-\infty}^{\infty} k(x, y) f(y)\, dy \tag{56}$$
then has a continuous spectrum (a nondenumerable infinity of eigenvalues) and, thus, T_k is not compact [7, p. 267]. The question arises: can we make use of such kernels in SV machines and still obtain generalization error bounds of the form developed above? Note that by a theorem of Widom [59], the eigenvalue decay of any convolution operator defined on a compact set via a kernel having compact support can decay no faster than $\lambda_j = \Omega(e^{-j^2})$ and thus if one seeks very rapid decay of eigenvalues (with concomitantly small entropy numbers), one must use convolution kernels with noncompact support.

We will resolve these issues in the present section. Before doing so, let us first consider the case that $\operatorname{supp} k \subseteq [-a, a]$ for some $a < \infty$. Suppose further that the data points $\boldsymbol{x}_j$ satisfy $\boldsymbol{x}_j \in [-b, b]$ for all j. If $k(\cdot, \cdot)$ is a convolution kernel (i.e., $k(x, y) = k(x-y, 0)$ which allows us to write with some abuse of notation $k(x-y) := k(x-y, 0)$), then the SV hypothesis $h_k(\cdot)$ can be written
$$h_k(x) := \sum_{j=1}^{m} \alpha_j k(x, \boldsymbol{x}_j) = \sum_{j=1}^{m} \alpha_j k_v(x, \boldsymbol{x}_j) =: h_{k_v}(x) \tag{57}$$
for $v \ge 2(a+b)$ where $k_v(\cdot)$ is the v-periodic extension of $k(\cdot)$ (analogously, $k_v(x-y) := k_v(x-y, 0)$)
$$k_v(x) := \sum_{j=-\infty}^{\infty} k(x - jv). \tag{58}$$
The d-dimensional Fourier transform is defined by
$$\begin{aligned}&F\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)\\ &F[f](\boldsymbol{\omega}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} f(\boldsymbol{x})\, d\boldsymbol{x}.\end{aligned} \tag{59}$$
Then, its inverse transform is given by
$$\begin{aligned}&F^{-1}\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)\\ &F^{-1}[f](\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} f(\boldsymbol{\omega})\, d\boldsymbol{\omega}.\end{aligned} \tag{60}$$
F can be shown to be an isometry on $L_2(\mathbb{R}^d)$.

We now relate the eigenvalues of T_{k_v} to the Fourier transform of $k(\cdot)$. We do so for the case of $d = 1$ and then state the general case later.

Lemma 14: Let $k\colon \mathbb{R} \to \mathbb{R}$ be a symmetric convolution kernel, let $K(\omega) = F[k](\omega)$ denote the Fourier transform of $k(\cdot)$, and k_v denote the v-periodical kernel derived from k (also assume that k_v exists). Then k_v has a representation as a Fourier series with $\omega_0 := \frac{2\pi}{v}$ and
$$\begin{aligned}k_v(x-y) &= \sum_{j=-\infty}^{\infty} \frac{\sqrt{2\pi}}{v} K(j\omega_0) e^{ij\omega_0 x}\\ &= \frac{\sqrt{2\pi}}{v} K(0) + \sum_{j=1}^{\infty} \frac{2}{v} \sqrt{2\pi} K(j\omega_0) \cos(j\omega_0(x-y)).\end{aligned} \tag{61}$$
Moreover, the eigenvalues λ_j of T_{k_v} satisfy $\lambda_j = \sqrt{2\pi} K(j\omega_0)$ for $j \in \mathbb{Z}$ and $C_k = \sqrt{\frac{2}{v}}$.

Proof: Clearly, the Fourier series coefficients K_j of k_v exist (as k_v exists) with
$$K_j := \frac{1}{\sqrt{v}} \int_{-v/2}^{v/2} e^{-ij\omega_0 x} k_v(x)\, dx$$

and therefore, by the definition of k_v and the existence of $K(\omega)$, we conclude

$$\begin{aligned} K_j &= \frac{1}{\sqrt{v}} \int_{-v/2}^{v/2} \sum_{j=-\infty}^{\infty} e^{-ij\omega_0 x} k(x - jv) \\ &= \frac{1}{\sqrt{v}} \sum_{j=-\infty}^{\infty} \int_{-v/2}^{v/2} e^{-ij\omega_0 x} k(x - jv) \\ &= \sqrt{\frac{2\pi}{v}}\, K(j\omega_0). \end{aligned}$$

This and the fact that

$$\{x \mapsto v^{-1/2} e^{ij\omega_0 x} : j \in \mathbb{Z}\}$$

forms an orthogonal basis in $L_2([-\frac{v}{2}, \frac{v}{2}], \mathbb{C})$ proves (61). (Note that since $k(x) = k(-x)$ we conclude $\overline{K(\omega)} = K(-\omega)$.) Furthermore, we are interested in real-valued basis functions for $k(x - y)$. The functions

$$\begin{aligned} &\psi_0(x) := \frac{1}{\sqrt{v}} \\ &\psi_j(x) := \sqrt{\frac{2}{v}}\, \cos(j\omega_0 x) \quad \text{and} \quad \psi_{-j}(x) := \sqrt{\frac{2}{v}}\, \sin(j\omega_0 x) \end{aligned} \tag{62}$$

for all $j \in \mathbb{N}$ satisfy $\|\psi_j\|_{L_2} = 1$, $j \in \mathbb{Z}$ and form an eigensystem of the integral operator defined by k_v with the corresponding eigenvalues $\sqrt{2\pi} K(j\omega_0)$. Finally, one can see that $C_k = \sqrt{\frac{2}{v}}$ by computing the max over $j \in \mathbb{N}$ and $x \in [-v/2, v/2]$. □

Thus, even though T_k may not be compact, T_{k_v} can be (if $(K(j\omega_0))_{j\in\mathbb{N}} \subset \ell_2$, for example). The above lemma can be applied whenever we can form $k_v(\cdot)$ from $k(\cdot)$. Clearly, $k(x) = O(x^{-(1+\epsilon)})$ for some $\epsilon > 0$ suffices to ensure the sum in (58) converges.

Let us now consider how to choose v. Note that the Riemann–Lebesgue lemma tells us that for integrable $k(\cdot)$ of bounded variation (surely any kernel one would use would satisfy that assumption), one has $K(\omega) = O(1/\omega)$. There is a tradeoff in choosing v in that for large enough ω, $K(\omega)$ is a decreasing function of ω (at least as fast as $1/\omega$) and, thus, by Lemma 14, $\lambda_j = \sqrt{2\pi} K(2\pi j/v)$ is an increasing function of v. This suggests one should choose a small value of v. But a small v will lead to high empirical error (as the kernel "wraps around" and its localization properties are lost) and large C_k. There are several approaches to picking a value of v. One obvious one is to *a priori* pick some $\tilde{\epsilon} > 0$ and choose the smallest v such that $|k(x) - k_v(x)| \leq \tilde{\epsilon}$ for all $x \in [-v/2, v/2]$. Thus, one would obtain a hypothesis $h_{k_v}(x)$ uniformly within $C\tilde{\epsilon}$ of $h_k(x)$ where $\sum_{j=1}^{m} |\alpha_j| \leq C$.

Remark 15: The above Lemma can be readily extended to d dimensions. Assume $k(\boldsymbol{x})$ is v-periodic in each direction ($\boldsymbol{x} = (x_1, \ldots, x_d)$), we get

$$\lambda_{\boldsymbol{j}} = (2\pi)^{\frac{d}{2}} K(\omega_0 \boldsymbol{j}) = (2\pi)^{\frac{d}{2}} K(\omega_0 \|\boldsymbol{j}\|) \tag{63}$$

for radially symmetric k and finally for the eigenfunctions $C_k = (2/v)^{\frac{d}{2}}$.

Finally, it is worth explicitly noting how the choice of a different bandwidth of the kernel, i.e., letting $k^{(\sigma)}(\boldsymbol{x}) := \sigma^d k(\sigma \boldsymbol{x})$, affects the eigenspectrum of the corresponding operator. We have $K^{(\sigma)}(\boldsymbol{\omega}) = K(\boldsymbol{\omega}/\sigma)$, hence scaling a kernel by σ means more densely spaced eigenvalues in the spectrum of the integral operator $T_{k^{(\sigma)}}$.

In conclusion, in order to obtain a discrete spectrum one needs to use a periodic kernel. For a given problem, one can always periodize a nonperiodic kernel in a way that changes the final hypothesis in an arbitrarily small way. One can then make use of the results of the present paper.

VII. Covering Numbers for Given Decay Rates

In this section, we will show how the asymptotic behavior of $\epsilon_n(A\colon \ell_2 \to \ell_2)$, where A is the scaling operator introduced before, depends on the eigenvalues of T_k.

A similar analysis has been carried out by Prosser [38], in order to compute the entropy numbers of integral operators. However, all of his operators mapped into $L_2(\mathcal{X}, \mathbb{C})$. Furthermore, while our propositions are stated as asymptotic results as his were, the proofs actually give nonasymptotic information with explicit constants.

Note that we need to sort the eigenvalues in a nonincreasing manner because of the requirements in Proposition 9. If the eigenvalues were unsorted one could obtain far too small numbers in the geometrical mean of $\lambda_1, \ldots, \lambda_j$. Many one-dimensional kernels have nondegenerate systems of eigenvalues in which case it is straightforward to explicitly compute the geometrical means of the eigenvalues as will be shown below. Note that while all of the examples below are for convolution kernels, i.e., $k(x, y) = k(x - y)$, there is nothing in the formulations of the propositions themselves that requires this. When we consider the d-dimensional case we shall see that with rotationally invariant kernels, degenerate systems of eigenvalues are generic. In Section VIII-B, we will show how to systematically deal with that case.

Let us consider the special case where $(\lambda_j)_j$ decays asymptotically with some polynomial or exponential degree. In this case, we can choose a sequence $(a_j)_j$ for which we can evaluate (40) explicitly. In what follows, by the eigenvalues of a kernel k we mean the (sorted) eigenvalues of the induced integral operator T_k.

Proposition 16 (Polynomial Decay): Let k be a Mercer kernel with eigenvalues $\lambda_j = O(j^{-(\alpha+1)})$ for some $\alpha > 0$. Then for any $\delta \in (0, \alpha/2)$ we have

$$\begin{aligned} \epsilon_n(A\colon \ell_2 \to \ell_2) &= O\left(\ln^{-\frac{\alpha}{2}+\delta} n\right) \\ \epsilon_n(A\colon \ell_2 \to \ell_2) &= \Omega\left(\ln^{-\frac{\alpha}{2}} n\right). \end{aligned}$$

An example of such a kernel is $k(x) = e^{-x}$. The proof can be found in Appendix I.

The next theorem covers a wide range of practically used kernels, namely, those with exponential polynomial decay in their

eigenvalues. For instance, the Gaussian kernel $k(x) = e^{-x^2}$ has exponential quadratic decay in λ_i. The "damped harmonic oscillator" kernel $k(x) = \frac{1}{1+x^2}$ is another example, this time with just exponential decay in its eigenvalues.

Proposition 17 (Exponential-Polynomial Decay): Suppose k is a Mercer kernel with $\lambda_j = O(e^{-\alpha j^p})$ for some $\alpha, p > 0$. Then

$$|\ln \epsilon_n(A\colon \ell_2 \to \ell_2)| = \Theta\left(\ln^{\frac{p}{p+1}} n\right). \tag{64}$$

See Appendix I for a proof. (A more precise, but rather more complex, calculation is given in [20].) While this theorem gives the guarantees on the learning rates of estimators using such types of kernels (which is theoretically pleasing and leads to desirable sample complexity rates), it may not always be wise to use the theoretically obtained bounds. Instead, one should take advantage of the estimates based on an analysis of the distribution of the training data since the rates obtained by the latter may turn out to be far superior with respect to the theoretical predictions (cf. Section VI and [61]).

VIII. Higher Dimensions

Things get somewhat more complicated in higher dimensions. For simplicity, we will restrict ourselves to translation-invariant kernels in what follows.

There are two simple ways to construct kernels in $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with $d > 1$. First one could construct kernels by

$$k(\boldsymbol{x} - \boldsymbol{y}) = k(x_1 - y_1) \times \cdots \times k(x_d - y_d). \tag{65}$$

This choice will usually lead to preferred directions in input space as the kernels are not rotationally invariant in general. The second approach consists in setting

$$k(\boldsymbol{x} - \boldsymbol{y}) = k(||\boldsymbol{x} - \boldsymbol{y}||_{\ell_2}). \tag{65}$$

This approach also leads to translationally invariant kernels which are also rotationally invariant. In the following, we will exploit this second approach to compute regularization operators and corresponding Green's functions. It is quite straightforward, however, to generalize our exposition to the rotationally asymmetric case. Now let us introduce the basic ingredients needed for the further calculations.

A. Basic Tools

Now introduce regularization operators P defined by

$$\langle Pf, Pg\rangle := \int_{\operatorname{supp} P(\boldsymbol{\omega})} \frac{\overline{F[f](\boldsymbol{\omega})} F[g](\boldsymbol{\omega})}{P(\boldsymbol{\omega})}\, d\boldsymbol{\omega} \tag{67}$$

for some nonnegative function $P(\boldsymbol{\omega})$ converging to 0 for $||\boldsymbol{\omega}|| \to \infty$. It can be shown [48] that for a kernel to be a Green's function of P^*P, i.e.,

$$\langle Pk(\boldsymbol{x}), Pk(\boldsymbol{x} - \boldsymbol{x}_0)\rangle = k(\boldsymbol{x}_0) \tag{68}$$

we need $F[k](\boldsymbol{\omega}) = P(\boldsymbol{\omega})$. For radially symmetric functions, i.e., $f(\boldsymbol{x}) = f(||\boldsymbol{x}||_2)$, we can explicitly carry out the integration on the sphere to obtain the Fourier transform which is also radially symmetric (see, e.g., [50, p. 33]), namely,

$$F[f](||\omega||) = \omega^{-\nu} H_\nu[r^\nu f(r)](||\omega||) \tag{69}$$

where $\nu := \frac{1}{2}d - 1$ and $H_\nu[\cdot]$ is the Hankel transform over the positive real line. The latter is defined by

$$H_\nu[f](\omega) := \int_0^\infty r f(r) J_\nu(\omega r)\, dr. \tag{70}$$

Here J_ν is the Bessel function of the first kind defined by

$$J_\nu(r) := r^\nu 2^{-\nu} \sum_{j=0}^\infty \frac{(-1)^j r^{2j}}{2^{2j} j! \Gamma(j + \nu + 1)}. \tag{71}$$

Note that $H_\nu = H_\nu^{-1}$, i.e., $f = H_\nu[H_\nu[f]]$ (in L_2) due to the Hankel inversion theorem [50].

B. Degenerate Systems

Computing the Fourier transform for a given kernel k gives us the continuous spectrum. As pointed out in Section VI, we are interested in the discrete spectrum of integral kernels defined on $\mathcal{X}$. This means that the eigenvalues are defined on the grid $\omega_0\mathbb{Z}^d$ with $\omega_0 = 2\pi/v$. Assuming $k(\boldsymbol{x})$ is rotationally invariant, so is $K(\boldsymbol{\omega})$ and, therefore, there are repeated eigenvalues $\lambda_{\boldsymbol{j}} = (2\pi)^{\frac{d}{2}} K(\boldsymbol{j}\omega_0)$. Consequently, we have degeneracies in the point spectrum of the integral operator given by k (or k_v, respectively) as all $\boldsymbol{j}\omega_0$ with equal length will have the same eigenvalue. In order to deal with this case efficiently we slightly modify Theorem 7 for our purposes. The following theorem allows proper account to be taken of the multiplicity of eigenvalues, and thus allows a more refined calculation of the desired entropy numbers.

Proposition 18: Let $(s_t)_t \in \mathbb{N}_0^{\mathbb{N}}$ be an increasing sequence with $s_0 = 1$ and $(\sigma_j)_j \in \mathbb{R}^{\mathbb{N}}$ be a nonincreasing sequence of nonnegative numbers with

$$\sigma_{s_j} < \sigma_{s_{\bar{j}}}, \qquad \text{for } j < \bar{j} \text{ and } \sigma_j = \sigma_{s_t} \text{ for } s_{t-1} < j \le s_t$$

and let

$$D\boldsymbol{x} = (\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_j x_j, \ldots) \tag{72}$$

for $\boldsymbol{x} = (x_1, x_2, \ldots, x_j, \ldots) \in \ell_p$ be the diagonal operator from ℓ_p into itself, generated by the sequence $(\sigma_j)_j$, where $1 \le p \le \infty$. Then for all $n \in \mathbb{N}$

$$\sup_{t\in\mathbb{N}} n^{-\frac{1}{s_t}} (\sigma_1\sigma_2 \cdots \sigma_{s_t})^{\frac{1}{s_t}} \le \epsilon_n(D) \le 6 \sup_{t\in\mathbb{N}} n^{-\frac{1}{s_t}} (\sigma_1\sigma_2 \cdots \sigma_{s_t})^{\frac{1}{s_t}}. \tag{73}$$

See Appendix II for a proof.

This proposition allows us to obtain a similar result to Proposition 9.

Proposition 19 (Degenerate Systems): Let $k\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be a Mercer kernel and let A be defined by (38) with the additional restriction that the coefficients a_j have to match the degeneracy

of λ_j, i.e., $a_{s_j} \geq a_{s_{\overline{j}}}$ for $j < \overline{j}$ and $a_j = a_{s_t}$ for $s_{t-1} < j \leq s_t$. Then one can choose A such that

$$\epsilon_n(A\colon \ell_2 \to \ell_2) \leq \inf_{(a_j)_j\colon (\sqrt{\lambda_j}/a_j)_j \in \ell_2} \sup_{t\in\mathbb{N}} 6C_k \times \left\| \left(\sqrt{\lambda_j}/a_j\right)_j \right\|_{\ell_2} n^{-\frac{1}{s_t}} (a_1 a_2 \cdots a_{s_t})^{\frac{1}{s_t}}. \quad (74)$$

This result by itself may not appear too useful. However, it is in fact exactly what we need for the degenerate case (it is slightly tighter than the original statement, as the supremum effectively has to be carried out only over a subset of $\mathbb{N}$). Finally, we have to compute the degree of multiplicity that occurs for different indexes $\boldsymbol{j}$. For this purpose, consider shells of radius r in $\mathbb{R}^d$ centered at the origin, i.e., rS^{d-1}, which contain a nonzero number of elements of $\mathbb{Z}^d$. Denote the corresponding radii by r_j and let $n(r_j, d)$ be the number of elements on these shells. Observe that $n(r, d) \neq 0$ only when $r^2 \in \mathbb{N}$. Thus

$$\begin{aligned} n(r, d) &:= |\mathbb{Z}^d \cap rS^{d-1}| \\ N(r, d) &:= \sum_{\{0\leq\rho\leq r\colon \rho^2\in\mathbb{N}\}} n(\rho, d). \end{aligned} \quad (75)$$

The determination of $n(r, d)$ is a classical problem which is completely solved by the use of the θ-series (see, e.g., [19]).

Theorem 20 (Occupation Numbers of Shells): Let the formal power series $\theta(x)$ be defined by

$$\theta(x) := \sum_{j=-\infty}^{\infty} x^{j^2} = 1 + 2\sum_{j=1}^{\infty} x^{j^2}. \quad (76)$$

Then

$$(\theta(x))^d = \sum_{j=1}^{\infty} n\left(\sqrt{j}, d\right) x^j. \quad (77)$$

This theorem allows one to readily compute $n(r, d)$ exactly; see Appendix IV for some Maple code to do so. (Note that while there do exist closed-form asymptotic approximate formulas for $n(r, d)$ [19, p. 155], they are inordinately complicated and of little use for our purposes.)

We can now construct an index of the eigenvalues which satisfies the required ordering (at least for nonincreasing functions $K(\omega)$) and we get the following result.

Proposition 21: Let $k\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be a Mercer kernel with eigenvalues given by a radially symmetric nonincreasing function on a lattice, i.e., $\lambda_{\boldsymbol{j}} = \lambda(\|\boldsymbol{j}\|)$ with $\boldsymbol{j} \in \mathbb{Z}^d$ and let A be defined by (38) with the additional restriction that the coefficients $a_{\boldsymbol{i}}$ have to match the degeneracy of $\lambda_{\boldsymbol{j}}$, i.e., $a_{\boldsymbol{j}} = a(\|\boldsymbol{j}\|)$. Then

$$\begin{aligned} &\epsilon_n(A\colon \ell_2 \to \ell_2) \\ &\leq \inf_{(a_{\boldsymbol{j}})_{\boldsymbol{j}}\colon \left(\frac{\sqrt{\lambda_{\boldsymbol{j}}}}{a_{\boldsymbol{j}}}\right)_{\boldsymbol{j}} \in (\ell_2)^d} \sup_{t\in\mathbb{N}} 6C_k \left\| \left(\frac{\sqrt{\lambda_{\boldsymbol{j}}}}{a_{\boldsymbol{j}}}\right)_{\boldsymbol{j}} \right\|_{(\ell_2)^d} \\ &\quad \times n^{-\frac{1}{N(r_t, d)}} \left(\prod_{q=1}^{t} a(r_q)^{n(r_q, d)}\right)^{\frac{1}{N(r_t, d)}}. \end{aligned} \quad (78)$$

Note that this result, although it may seem straightforward, cannot be obtained from Proposition 9 directly as there the sup would have to be carried out over $\mathbb{N}$ instead of $(N(r_t, d))_t$.

C. Bounds for Kernels in $\mathbb{R}^d$

Let us conclude this section with some examples of the eigenvalue sequences for kernels typically used in SV machines. These can then be used to evaluate the right-hand side in Corollary 21. Recall that $\nu = (\frac{d}{2}) - 1$. First we have to compute the Fourier/Hankel transform for the kernels.

Example 22 (Gaussian RBFs): For Gaussian radial basis functions (RBFs) in d dimensions we have $k(r) = \sigma^{-d} e^{-\frac{r^2}{2\sigma^2}}$ and correspondingly

$$\begin{aligned} F[k](\omega) &= \omega^{-\nu}\sigma^{-d} H_\nu \left[r^\nu e^{-\frac{r^2}{2\sigma^2}}\right](\omega) \\ &= \omega^{-\nu}\sigma^{2(\nu+1)-d}\omega^\nu e^{-\frac{\omega^2\sigma^2}{2}} \\ &= e^{-\frac{\omega^2\sigma^2}{2}}. \end{aligned}$$

Example 23 (Exponential RBFs): In the case of $k(r) = e^{-ar}$ we obtain

$$\begin{aligned} F[k](\omega) &= \omega^{-\nu} H_\nu \left[r^\nu e^{-ar}\right](\omega) \\ &= \omega^{-\nu} 2^{\nu+1}\omega^\nu a\pi^{-\frac{1}{2}}\Gamma(\nu+\tfrac{3}{2})(a^2+\omega^2)^{-\nu-\frac{3}{2}} \\ &= 2^{d/2} a\pi^{-\frac{1}{2}}\Gamma\left(\frac{d}{2}+1\right)\frac{1}{(a^2+\omega^2)^{\frac{d+1}{2}}} \end{aligned}$$

i.e., in the case of $d = 1$ we recover the damped harmonic oscillator (in the frequency domain). In general, we get a decay in terms of the eigenvalues like $\omega^{-(d+1)}$. Moreover, we can conclude from this that the Fourier transform of k, viewed itself as a kernel, i.e., $k(r) = (1+r^2)^{-\frac{d+1}{2}}$, yields the initial kernel as its corresponding power spectrum in Fourier domain.

Example 24 (Damped Harmonic Oscillator): Another way to generalize the harmonic oscillator, this time in a way that k does not depend on the dimensionality d, is to set $k(r) = \frac{1}{a^2+r^2}$. Following [58, Sec. 13.6] we get

$$\begin{aligned} F[k](\omega) &= \omega^{-\nu} H_\nu \left[\frac{r^\nu}{a^2+r^2}\right](\omega) \\ &= \omega^{-\nu} a^\nu K_\nu(\omega a) \end{aligned}$$

where K_ν is the Bessel function of the second kind, defined by (see [50])

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t}\cosh(\nu t)\, dt. \quad (79)$$

It is possible to upper-bound $F[k]$ by utilizing the asymptotic representation

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x} \quad (80)$$

(see, for example, [18, eq. (8.451.6)]) and we get exponential decay of the eigenvalues.

Using Theorem 20, Corollary 21, and Remark 15 one may compute the entropy numbers numerically for a particular kernel and a particular set of parameters. This may seem unsatisfactory

from a theoretician's point of view. However, as the ultimate goal is to use the obtained bounds for model selection, it is desirable to obtain as tight bounds (especially in the constants) as possible. Hence, if much more precise bounds can be obtained by some not too expensive numerical calculation it is definitely worthwhile to use those instead of a theoretically nice but not sufficiently tight upper bound. The computational effort to calculate these quantities is typically negligible in comparison to training the actual learning machine.

Notwithstanding the above, in order to give a feeling for the effect of the decay of the Fourier transform of the kernel on the entropy numbers of the A operator, we conclude with the following general result, the proof of which is in Appendix III.

Proposition 25 (Polynomial Exponential Decay in $\mathbb{R}^d$): For kernels $k(\cdot,\cdot)$ in $\mathbb{R}^d \times \mathbb{R}^d$ with $\lambda(\omega) = O(e^{-\alpha\|\omega\|^p})$ with $\alpha, p > 0$ the entropy number of the corresponding scaling operator satisfies

$$|\ln \epsilon_n(A\colon \ell_2 \to \ell_2)| = O\left(\ln^{\frac{p}{p+d}} n\right).$$

IX. Conclusion

We have shown how to connect properties known about mappings into feature spaces with bounds on the covering numbers. Exploiting the geometric structure of the feature-space map enabled us to relate the properties of the kernel inducing the feature space to the covering numbers of the class of functions implemented by SV machines based on such kernels.

The actual application of our results, perhaps for model selection using structural risk minimization, is somewhat involved, but is certainly doable. Here, we outline one possible path. In [20] we present an application of the results to the performance of SV machines for pattern classification.

A. One Way to Use the Results of this Paper

Choose k and σ: The kernel k may be chosen for a variety of reasons, which we have nothing additional to say about here. The choice of σ should take account of the discussion in Section VI.

Choose the Period v of the Kernel: One suggested procedure is outlined in Section VI.

Bound $\epsilon_n(A)$: This can be done using Proposition 9 (for the case $d = 1$) or Corollary 19 or 21 for the case $d > 1$. Some examples of this sort of calculation are given in Section VII.

Bound $\epsilon_n(T)$: Using Theorem 12.

Take Account of the "$+B$": The key observation is that given a class $\mathcal{F}$ with known $\mathcal{N}^m(\epsilon, \mathcal{F})$, one can bound $\mathcal{N}^m(\epsilon, \mathcal{F}^+)$ as follows. (Here $\mathcal{F}^+ := \{f + b\colon f \in \mathcal{F}, b \in \mathbb{R}\}$.) Suppose V_ϵ is an ϵ-cover for $\mathcal{F}$ and elements of $\mathcal{F}^+$ are uniformly bounded by B (this implies a limit on $|b|$ as well as a uniform bound on elements of $\mathcal{F}$). Then

$$V_\epsilon^+ := \bigcup_{j=-B/\epsilon}^{B/\epsilon} V_\epsilon + j\epsilon$$

is an ϵ-cover for $\mathcal{F}^+$ and thus

$$\mathcal{N}^m(\epsilon, \mathcal{F}^+) \le \frac{2B}{\epsilon}\mathcal{N}^m(\epsilon, \mathcal{F}).$$

Observe that this will only be "noticeable" for classes $\mathcal{F}$ with very slowly growing covering numbers (polynomial in $1/\epsilon$).

Take Account of the Loss Function: Using Lemma 2 for example.

Plug into a Uniform Convergence Result: See the pointers to the literature and the example in Section IV.

B. Further Work

The operator-theoretic viewpoint introduced in this paper seems fruitful. The overall bounds for SV classes can, via a somewhat involved argument, be considerably simplified [20]. The general approach can be applied to various other learning machines such as convex combinations of basis functions and multilayer networks [47]. When combined with an appropriate statistical argument [45], the approach yields bounds on the generalization that depend strongly on the particular sample observed [61]. The methods can also be applied to some problems of unsupervised learning [49].

The results of the present paper hinge on the measurement of the size of the weight vector $\boldsymbol{w}$ by an ℓ_2 norm. In [62], we show the effect of different norms for measuring the size of $\boldsymbol{w}$, as well as presenting a number of related results.

We expect that further refinements and extensions to these techniques will continue to yield interesting results.

Appendix I
Proofs of Results in Section VII

Proof (Proposition 16): The proof uses Proposition 9. Since $\lambda_j = O(j^{-\alpha-1})$ there exists some $\beta \in \mathbb{R}^+$ with $\lambda_j \le \beta^2 j^{-\alpha-1}$. In this case, all sequences $(a_j)_j = (j^{-\frac{\tau}{2}})_j$ with $0 < \tau < \alpha$ lead to an admissible scaling property. One has

$$\left\|\left(\frac{\sqrt{\lambda_j}}{a_j}\right)_j\right\|_{\ell_2} = \beta\left\|\left(j^{\frac{\tau-\alpha-2}{2}}\right)_j\right\|_{\ell_2} = \beta\sqrt{\zeta(\alpha-\tau+1)} \tag{81}$$

where $\zeta(\cdot)$ is Riemann's zeta function. Moreover, one can bound $\zeta(\cdot)$ by

$$x + \gamma \le \zeta\left(1 + \frac{1}{x}\right) \le x + 1 \tag{82}$$

where γ is Euler's constant. The next step is to evaluate the expression

$$(a_1 a_2 \cdots a_j)^{\frac{1}{j}} = \left(\prod_{s=1}^{j} s^{-\frac{\tau}{2}}\right)^{\frac{1}{j}} = (j!)^{-\frac{\tau}{2j}} = \Gamma(j+1)^{-\frac{\tau}{2j}}. \tag{83}$$

The Gamma function $\Gamma(x)$ can be bounded as follows: for $j > 1$

$$\ln j - 1 \le \frac{1}{j}\ln\Gamma(j+1) \le \ln j. \tag{84}$$

Hence, one may bound $\epsilon_n(A)$

$$\epsilon_n(A) \geq C_k\beta \inf_{\tau\in(0,\alpha)} \sup_{j\in\mathbb{N}} n^{-\frac{1}{j}} \left(\frac{1}{\alpha-\tau}+\gamma\right)^{\frac{1}{2}} j^{-\frac{\tau}{2}}$$

$$\epsilon_n(A) \leq 6C_k\beta \inf_{\tau\in(0,\alpha)} \sup_{j\in\mathbb{N}} n^{-\frac{1}{j}} \left(\frac{1}{\alpha-\tau}+1\right)^{\frac{1}{2}} e^{\frac{\tau}{2}} j^{-\frac{\tau}{2}}.$$

In order to avoid unneeded technicalities we will replace $\sup_{j\in\mathbb{N}}$ by $\sup_{j\in[1,\infty)}$. This is no problem when computing the upper bound, but it is an issue for the lower bound. However, $j^{-\frac{\tau}{2}}$ on $[1,\infty)$ is within a constant factor of $2^{-\frac{\tau}{2}}$ of its corresponding values on the integer domain $\mathbb{N}$, the biggest discrepancy being at $[1, 2]$.[1] Thus, we may safely ignore the concern. Next we compute

$$\sup_{j\in[1,\infty)} n^{-\frac{1}{j}} j^{-\frac{\tau}{2}} = \sup_{j\in[1,\infty)} e^{-\frac{1}{j}\ln n - \frac{\tau}{2}\ln j} = \left(\frac{2e\ln n}{\tau}\right)^{-\frac{\tau}{2}}. \tag{85}$$

The maximum of the argument is obtained for $j=\frac{2\ln n}{\tau}$, hence (85) holds for all $\ln n \geq \frac{\tau}{2}$, which is fine since we want to compute bounds on $\epsilon_n(A)$ as $n\to\infty$. For the lower bounds on $\epsilon_n(A)$ we obtain

$$\begin{aligned}\epsilon_n(A) &\geq C_k\beta(2e)^{-\frac{\tau}{2}} \inf_{\tau\in(0,\alpha)} \left(\frac{1}{\alpha-\tau}+\gamma\right)^{\frac{1}{2}} \left(\frac{2\ln n}{\tau}\right)^{-\frac{\tau}{2}} \\ &\geq C_k\beta(2e)2^{-\frac{\tau}{2}} \inf_{\tau\in(0,\alpha)} \left(\frac{1}{\alpha-\tau}+\gamma\right)^{\frac{1}{2}} \\ &\quad\cdot \inf_{\tau\in(0,\alpha)} \left(\frac{2\ln n}{\tau}\right)^{-\frac{\tau}{2}} \\ &= C_k\beta(2e)2^{-\frac{\tau}{2}} \left(\frac{1}{\alpha}+\gamma\right)^{\frac{1}{2}} \left(\frac{2\ln n}{\alpha}\right)^{-\frac{\alpha}{2}}. \end{aligned} \tag{86}$$

This shows that $\epsilon_n(A)$ is always bounded from below by $\Omega(\ln^{-\frac{\alpha}{2}} n)$. Computation of the upper bound is slightly more effort, since one has to evaluate

$$\epsilon_n(A) \leq 6C_k\beta \inf_{\tau\in(0,\alpha)} \left(\frac{1}{\alpha-\tau}+1\right)^{\frac{1}{2}} \left(\frac{2\ln n}{\tau}\right)^{-\frac{\tau}{2}}. \tag{87}$$

Clearly, for any fixed $\tau\in(0,\alpha)$ we are able to obtain a rate of $\epsilon_n(A)=O(\ln^{-\frac{\tau}{2}} n)$, thus, the theorem follows. For practical purposes, a good approximation of the inf can be found as $\frac{1}{\alpha-\tau}=\ln(2\ln n)$ by computing the derivative of the argument in (87) with respect to τ and dropping all terms independent of τ and n. However, numerical minimization of (87) is more advisable when small values of $\epsilon_n(A)$ are crucial. □

For the proof of Proposition 17 we need the following standard Lemma.

[1]One may show [61] that

$$a_{j^*+1} \leq \sup_{j\in\mathbb{N}} n^{-\frac{1}{j}} (a_1,\ldots,a_j)^{\frac{1}{j}} \leq a_{j^*}$$

for that particular j^* where $\sup_{j\in\mathbb{N}}$ is actually obtained. Hence, the maximum quotient a_{j+1}/a_j, which in the present case is $2^{-\frac{\tau}{2}}$, determines the value by which the bound has to be lowered in order to obtain a true lower bound.

Lemma (Summation and Integration in $\mathbb{R}$): Suppose $f\colon\mathbb{R}\to\mathbb{R}$ is an integrable nonincreasing function. Then the following inequality holds for any $a\in\mathbb{Z}$:

$$\int_a^\infty f(x)\,dx \leq \sum_{n=a}^\infty f(n) \leq \int_{a-1}^\infty f(x)\,dx. \tag{88}$$

Proof: The proof relies on the fact that

$$f(n) \geq \int_n^{n+1} f(n)\,dn \geq f(n+1)$$

due to the monotonicity of f and a decomposition of the integral

$$\int_0^\infty = \sum_{n=0}^\infty \int_n^{n+1}.$$

The lemma is a direct consequence thereof. □

Proof (Proposition 17): Since $\lambda_j = O(e^{-\alpha j^p})$ there exists some $\beta\in\mathbb{R}^+$ with $\lambda_j \leq \beta^2 e^{-\alpha j^p}$. Similarly as before, we now use a series $(a_j)_j = e^{-\tau/2j^p}$. Then by applying Lemma 26 we have that for any $\tau\in[0,\alpha)$

$$\left\| \left(\frac{\sqrt{\lambda_j}}{a_j}\right)_j \right\|_{\ell_2} = \beta \left(\sum_{j=0}^\infty e^{(\tau-\alpha)j^p}\right)^{\frac{1}{2}} \begin{cases} \leq \beta\sqrt{1+\frac{\Gamma(1/p)}{p(\alpha-\tau)^{1/p}}} \\ \geq \beta\sqrt{\frac{\Gamma(1/p)}{p(\alpha-\tau)^{1/p}}}. \end{cases} \tag{89}$$

Next, we have to apply a similar bound to the product of the first j diagonal entries of the scaling operator A

$$(a_1a_2\cdots a_j)^{\frac{1}{j}} = e^{-\frac{1}{2j}\tau\sum_{s=1}^j s^p} \begin{cases} \geq e^{-\frac{\tau}{2(p+1)}j^p} \\ \leq e^{-\frac{\tau}{2(p+1)}j^p + \frac{\tau}{2j(p+1)}} \\ \leq e^{-\frac{\tau}{2(p+1)}j^p + \frac{\tau}{2(p+1)}}. \end{cases} \tag{90}$$

The last inequality holds since $j\in\mathbb{N}$. Next we compute

$$\sup_{j\in\mathbb{N}} n^{-\frac{1}{j}} e^{-\frac{\tau}{2(p+1)}j^p} = \sup_{j\in\mathbb{N}} e^{-\frac{1}{j}\ln n - \frac{\tau}{2(p+1)}j^p}.$$

Differentiation of the exponent with respect to j leads to

$$j^{-2}\ln n - \frac{\tau p}{2(p+1)} j^{p-1} = 0 \Rightarrow j^{-1}\ln n = \frac{\tau p}{2(p+1)} j^p \tag{91}$$

and, thus,

$$\sup_{j\in[1,\infty)} n^{-\frac{1}{j}} e^{-\frac{\tau}{2(p+1)}j^p} = e^{-(\frac{\tau}{2})^{1/(p+1)}} \left(\frac{p+1}{p}\ln n\right)^{\frac{p}{p+1}}. \tag{92}$$

Replacing the domain from $\sup_{j\in\mathbb{N}}$ to $\sup_{j\in[1,\infty)}$ is not a problem when it comes to computing upper bounds on $\epsilon_n(A)$. As for the lower bounds, again, a similar reasoning to that in the

previous proof would have to be applied.[2] (The proof is omitted here.) Thus, $\epsilon_n(A)$ can be bounded from below as follows:

$$\begin{aligned}\epsilon_n(A) &\geq C_k\beta \inf_{\tau\in(0,\,\alpha)} \sqrt{\frac{\Gamma(1/p)}{p(\alpha-\tau)^{1/p}}} \\ &\quad \times e^{-(\frac{\tau}{2})^{1/(p+1)}} \left(\frac{p+1}{p}\ln n\right)^{\frac{p}{p+1}} \\ &\geq C_k\beta \inf_{\tau\in(0,\,\alpha)} \sqrt{\frac{\Gamma(1/p)}{p(\alpha-\tau)^{1/p}}} \inf_{\tau\in(0,\,\alpha)} \\ &\quad \cdot e^{-(\frac{\tau}{2})^{1/(p+1)}} \left(\frac{p+1}{p}\ln n\right)^{\frac{p}{p+1}} \\ &= C_k\beta \sqrt{\frac{\Gamma(1/p)}{p\alpha^{1/p}}}\, e^{-(\frac{\alpha}{2})^{1/(p+1)}} \left(\frac{p+1}{p}\ln n\right)^{\frac{p}{p+1}}.\end{aligned} \tag{93}$$

Hence, a lower bound on the rate of $\log \epsilon_n$ is $\Omega(\log^{\frac{p}{p+1}} n)$. Moreover, for the upper bound we obtain

$$\begin{aligned}\epsilon_n(A) \leq 6C_k\beta \inf_{\tau\in(0,\,\alpha)} &\sqrt{1+\frac{\Gamma(1/p)}{p(\alpha-\tau)^{1/p}}} \\ &e^{-(\frac{\tau}{2})^{1/(p+1)}} \left(\frac{p+1}{p}\ln n\right)^{\frac{p}{p+1}} + \frac{\tau}{2j(p+1)}.\end{aligned} \tag{94}$$

One could evaluate (94) numerically. However, it can be seen that for any fixed $\tau \in (0, \alpha)$ the rate of $\log \epsilon_n(A)$ can be bounded by $O(\log^{\frac{p}{p+1}} n)$, which shows that the obtained rates are tight. □

Appendix II
Proof of Theorem 18

Proof: The first part of the inequality follows directly from Theorem 7 as it is a weaker statement than the original one. We prove the second part by closely mimicking the proof in [13, p. 17]. We define

$$\delta(n) := 8 \sup_{t\in\mathbb{N}} n^{-\frac{1}{s_t}} (\sigma_1\sigma_2\cdots\sigma_{s_t})^{\frac{1}{s_t}} \tag{95}$$

and show that for all n there is an index s_j with $\sigma_{s_j+1} \leq \frac{\delta(n)}{4}$. For this purpose, choose an index r such that $n \leq 2^{s_j+1}$ and, thus, $1 \leq 2n^{-1/(s_j+1)}$. Moreover, we have

$$\sigma_{s_j+1} \leq (\sigma_1\sigma_2\cdots\sigma_{s_j+1})^{\frac{1}{s_j+1}} \tag{96}$$

because of the monotonicity of $(\sigma_j)_j$ and, finally,

$$\sigma_{s_j+1} \leq 2n^{-1/(s_j+1)} (\sigma_1\sigma_2\cdots\sigma_{s_j+1})^{\frac{1}{s_j+1}}. \tag{97}$$

Using the definition of $\delta(n)$ we thus conclude $\sigma_{s_j+1} \leq \delta(n)/4$. If this happens to be the case for σ_1, we have $\epsilon_n(D) \leq \sigma_1$ which proves the theorem.

[2]As in the previous theorem, the problem reduces to bounding the quotient a_{j^*+1}/a_{j^*} where j^* is the variable for which $\sup_{j\in\mathbb{N}}$ is obtained. However, here the quotient can only be bounded by $e^{-\frac{\tau p}{2} j^{p-1}}$. Fortunately, this is of lower order than the remaining terms, hence it will not change the *rate* of the lower bounds.

If this is not the case, there exists an index s_j such that $\sigma_{s_j+1} \leq \delta(n)/4 < \sigma_{s_j}$. Hence the corresponding sectional operator

$$\begin{aligned}&D_{s_j}\colon \ell_p \to \ell_p \quad \text{with} \\ &D_{s_j}(x_1, x_2, \ldots, x_{s_j}, x_{s_j+1}, \ldots) \\ &\quad = (\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_{s_j} x_{s_j}, 0, 0, \ldots)\end{aligned} \tag{98}$$

is of rank s_j and the image $D_{s_j}(U_p)$ of the closed unit ball U_p of ℓ_p is isometric to the subset $D^{(s_j)}(U_p^{(s_j)})$ of $\ell_p^{s_j}$. In any case, $D_{s_j}(U_p)$ is a precompact subset of ℓ_p. So let $y_1, y_2, \ldots, y_N$ be a maximal system of elements in $D_{s_j}(U_p)$ with

$$\|y_j - y_{\overline{j}}\| > \delta(n)/2, \qquad \text{for } j \neq \overline{j}. \tag{99}$$

The maximality of this system guarantees that

$$D_{s_j}(U_p) \subseteq \bigcup_{j=1}^{N} \left\{ y_j + \frac{\delta(n)}{2} U_p \right\} \tag{100}$$

and, thus, $\epsilon_N(D_{s_j}) \leq \delta(n)/2$. In order to get an estimate for $\epsilon_N(D)$, we split the operator D into two parts $D = (D - D_{s_j}) + D_{s_j}$ which allows us to bound

$$\epsilon_N(D) \leq \|D - D_{s_j}\| + \epsilon_N(D_{s_j}). \tag{101}$$

Using $\|D - D_{s_j}\| = \sigma_{s_j+1} \leq \delta(n)/4$ and the bound on $\epsilon_N(D_{s_j})$ we arrive at

$$\epsilon_N(D) \leq \frac{3}{4}\,\delta(n). \tag{102}$$

The final step is to show that $N \leq n$ as then by substituting in the definition of $\delta(n)$ into (102) yields the result. This is again achieved by a comparison of volumes. Consider the sets $\{y_j + (\delta(n)/4)U_p^{s_j}\}$ as subsets of the space $\ell_p^{s_j}$ which is possible since $y_j \in D_{s_j}(U_p)$ and $D_{s_j}(U_p) = D^{(s_j)}(U_p^{s_j})$. These sets are obviously pairwise-disjoint. On the other hand, we have

$$\bigcup_{j=1}^{N} \left\{ y_j + \frac{\delta(n)}{4} U_p^{s_j} \right\} \subseteq D^{(s_j)}(U_p^{s_j}) + \frac{\delta(n)}{4} U_p^{s_j}) \subseteq 2D^{(s_j)}(U_p^{s_j}) \tag{103}$$

as $\delta(n)/4 < \sigma_1$. Now a comparison of the d-dimensional Euclidean volumes vol_d provides

$$N \left(\frac{\delta(n)}{4}\right)^{s_j} \mathrm{vol}_{s_j}(U_p^{s_j}) \leq 2^{s_j}\sigma_1\sigma_2\cdots\sigma_{s_j} \mathrm{vol}_{s_j}(U_p^{s_j}) \tag{104}$$

and, therefore, $N \leq (8/\delta(n))^{s_j}\sigma_1\sigma_2\cdots\sigma_{s_j}$. Using the definition of $\delta(n)$ this yields $N \leq n$. □

Appendix III
Proof of Proposition 25

Proof: We will completely ignore the fact that we are actually dealing with a countable set of eigenvalues on a lattice and replace all summations by integrals without further worry. Of course this is not accurate but still will give us the correct rates for the entropy numbers.

Denote $1/\upsilon := (2\pi/v)^{\frac{d}{2}}$ the size of a unit cell, i.e., $\upsilon = (v/(2\pi))^{\frac{d}{2}}$ the density of lattice points in frequency space as

given in Section VI. Then we get for infinitesimal volumes dV and numbers of points dN in frequency space

$$dV = S_{d-1} r^{d-1}\, dr \quad \text{and, therefore, } dN = \upsilon S_{d-1} r^{d-1}\, dr \tag{105}$$

(here S_{d-1} denotes the volume of the $d-1$-dimensional unit sphere) leading to

$$N(r, d) = \frac{1}{d}\, \upsilon S_{d-1} r^d. \tag{106}$$

We introduce a scaling operator whose eigenvalues decay like $a(\omega) = e^{-\frac{\tau}{2}\|\omega\|^p}$ for $\tau \in [0, \alpha)$. It is straightforward to check that all these values lead to both useful and admissible scaling operators. Now we will estimate the separate terms in (78).

$$\begin{aligned}\left\| \left(\frac{\sqrt{\lambda_{\boldsymbol{i}}}}{a_{\boldsymbol{i}}} \right)_{\boldsymbol{i}} \right\|_{\ell_2}^2 &\approx \int dN(\omega)\, \frac{\lambda(\omega)}{a^2(\omega)} \\ &= S_{d-1}\upsilon \int_0^\infty r^{d-1} \beta^2 e^{-(\alpha-\tau)\|\omega\|^p} \\ &= S_{d-1}\upsilon\beta^2 (\alpha-\tau)^{-\frac{d}{p}} \tau \left(\frac{d}{p}\right) p^{-1}. \end{aligned} \tag{107}$$

Next, we have

$$\ln\left(n^{-\frac{1}{N(r,d)}} \right) = -\frac{d}{\upsilon S_{d-1} r_0^d}\, \ln n \tag{108}$$

and

$$\begin{aligned}\ln(a_1 \cdot a_2 \cdots a_{N(r,d)})^{\frac{1}{N(r,d)}} &= -\frac{d}{\upsilon S_{d-1} r_0^d} \sum_{j=1}^{N(r,d)} \ln a_j \\ &\approx d r^{-d} \int_0^r \omega^{d-1} \ln a(\omega)\, d\omega \qquad (109) \\ &= -d r^{-d} \int_0^r \omega^{d-1} \frac{\tau}{2}\, \omega^p\, d\omega \\ &= -\frac{\tau}{2}\, \frac{d}{d+p}\, r^p. \qquad (110)\end{aligned}$$

This leads to

$$\epsilon_n \le 6 C_k \beta \sqrt{\frac{S_{d-1}\upsilon\Gamma\left(\frac{d}{p}\right)}{p}} \inf_{\tau\in[0,\alpha)} (\alpha-\tau)^{-\frac{d}{2p}} \sup_{r\in\mathbb{R}^+} \exp\left(-\frac{d}{\upsilon S_{d-1} r^d} \ln n - \frac{\tau}{2}\, \frac{d}{d+p}\, r^p \right). \tag{111}$$

Computing the $\sup_{r\in\mathbb{R}^+}$ yields

$$r = \left(\frac{2}{\tau \upsilon S_{d-1}}\, \frac{(d+p)d}{p} \ln n \right)^{\frac{1}{d+p}} \tag{112}$$

and, therefore,

$$\epsilon_n \le 6 C_k \beta \sqrt{\frac{S_{d-1}\upsilon\Gamma\left(\frac{d}{p}\right)}{p}} \inf_{\tau\in[0,\alpha)} (\alpha-\tau)^{-\frac{d}{2p}} \exp\left(-\left(\frac{\tau}{2}\right)^{\frac{d}{d+p}} \left(\frac{(d+p)d}{p}\, \frac{\ln n}{\upsilon S_{d-1}} \right)^{\frac{p}{d+p}} \right). \tag{113}$$

Already from this expression one can observe the rate bounds on ϵ_n. What remains to be done is to compute the infimum over τ. This can be done by differentiating (113) with respect to τ. Define

$$T_n := \left(\frac{(d+p)d}{p}\, \frac{\ln n}{\upsilon S_{d-1}} \right)^{\frac{p}{d+p}} \tag{114}$$

which leads to the optimality condition on τ

$$(\alpha-\tau)\tau^{-\frac{p}{d+p}} = \frac{d+p}{2T_n p}\, 2^{\frac{d}{d+p}}, \qquad \text{with } \tau \in (0, \alpha] \tag{115}$$

which can be solved numerically. □

APPENDIX IV
MAPLE CODE TO COMPUTE $n(r, d)$

The following function can be used to compute $n(r, d)$: $t(m, d) = n(m^2, d)$.

```
# This code defines a function t where
# t(m,d) is number of points on a sphere of
# radius^2=m from Z^d
h:=n-> eval('if'(isolve(m^2=n,m)=NULL,0,
       'if'(n=0,1,2)),1):
powseries[powcreate](theta(n)=h(n)): t:=(m,d)->
coeff(convert(powseries[tpsform](
        powseries[evalpow](theta^d),
        x,m+1),polynom),x,m):
```

Note Added in Proof

Steve Smale has pointed out to us that Claim 2 of Theorem 4 due to König is false. This causes some of the intermediate results in the paper to be false, but not the main theorems. One can get around the false result by redefining

$$C_k := \sup_j \sup_{x\in\mathcal{X}} |\psi_j(x)|.$$

(Note that most practically used kernels still have $C_k < \infty$). Only the "if" claim of Proposition 6 remains true (all we need for or bounds on entropy numbers anyway). All of the upper bounds on entropy numbers still hold as long as C_k (redefined as above) is finite. A detailed correction can be found in Chapter 12 of B. Schölkopf and A. Smola, *Learning with Kernels*, Cambridge, MA: MIT Press, 2001.

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